Compactification of the moduli space in symplectization and hidden symmetries of its boundary

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1 Introduction

The purpose of this paper and the forth coming [L1], [L3] is to lay down a foundation for a sequence of papers concerning the moduli space of connecting pseudo-holomorphic maps in the symplectization of a compact contact manifold and their applications. In this paper, we will establish the comapctification of the moduli space of the pseudo-holomorphic maps in the symplectization and exhibit some new phenomenon concerning bubbling and the "hidden" symmetries of the boundary of the comapctification. Combining with the index formula, which will be proved in [L3], we will show in [L3] that the virtual co-dimension of the boundary components of the moduli space with at least one bubble is at least two, while the virtual co-dimension of the boundary components of broken connecting maps of two elements is one. In [L1], we will show that these virtual co-dimensions can be realized in the corresponding virtual moduli cycles. In a sequence of forth coming papers, we will give some of possible applications. In particular, we will define various versions of index homology for a contact manifold, relative index homology for a symplectic manifold with contact type boundary, as well as their multiplicative structures in these holmologies. These multiplicative structures can be thought as analogies of the usual quantum product and pants product in quantum cohomology and Floer cohomology. We will also investigate the implication of these homologies to Weinstein conjecture.

It is well-known that a family of pseudo-holomorphic maps in the symplectization of a compact contact manifold may develop bubbles. Since in the symplectization the symplectic form is exact, each top bubble necessarily has non-removable singularity at infinity, and along the end at infinity, the bubble is convergent to some closed orbit of the Reeb field of the contact manifold. This makes the behavior of the boundary components of the compactification of the moduli space here very much look like the one of the broken connecting orbits in the usual Floer homology. In particular, it is believed that the co-dimension of the boundary components even coming from bubbling should be one in general. We will show in this paper and [L3] that in the case of the moduli space the pseudo-holomorphic maps connecting at least two closed orbits at the two ends of the symplectization,at least virtually, this belief is not true.

Our starting point is the the following new phenomenon concerning the bubbling of connecting pseudo-holomorphic maps. Observe that each time when a family of pseudo-holomorphic maps connecting two closed orbits splits into a family of broken connecting maps or develops a bubble, there is not only a splitting of the domain but also a splitting of the target at same time. Therefore the **R**-symmetry of the target splits into a two-dimensional or multi-dimensional symmetries during the bubbling or splitting. Moreover, the rates of these two types of degeneration of the domain and target are independent to each other in general. In fact, the maximum principal implies that in the simplest case when such a family of connecting maps develops only one bubble, the image of the bubble lies on a new component on the "left" of the original one, and there is also a new principal component on the left of the original principal component. Note that the new "left" principal componet may be just a trivial connecting map. However, the limit map itself is still stable. This last kind of degeneration plays a rather special role. Therefore, unlike the usual Gromov-Floer theory in symplectic case, the bubbling here, splits the domain into three components and the target into two. Note that this phenomenon can only happen when the pseudo-holomorphic maps involved connect at least two closed orbits lying on the two ends of the symplectization.

Now using the fact that both symmetry groups of a connecting map and a bubble with non-removable singularity are three dimensional, it is easy to see that in term of the dimensions of symmetries, bubbling has co-dimension three, while the splitting of connecting maps into broken ones is co-dimension two. This seems to suggest a rather different picture on the boundary behavior of the moduli space in the symplectization, which is not only disprove what was believed before but also bring us to a situation of dilemma. Namely, the situation here is even better than the one in the usual Gromov-Floer theory in symplectic case.

One the the main purpose of this paper and [L1], [L3] is to resolve this dilemma. In this paper, we will give some key ingradients of the solution of the dilemma. The main body of this paper is devoted to to define the notion of stable maps in the symplectization and to use them to establish the compactness of the moduli space of such maps. It turns out situation here is different from the usual Gromov-Floer theory. There are various new phenomenons, which have to be put into consideration in order to to formulate the notion of stable maps and various related notions.

In symplectic geometry, one of the key ingredients to prove the compactness of the moduli space of stable maps is the bubbling process. It consists of three parts: the Uhlenbeck-Sacks rescaling scheme, removable singularity lemma and the analysis concerning the behavior of the "connecting tubes". In the case of the symplectization of a contact manifold, we have mentioned above that there is a new phenomenon in the bubbling process. However, as far as the proof goes, there are still the correspondoing three parts there. The first and most important part of the bubbling was established by Hofer in [H]. He discovered the phenomenon of bubbling in the symplecitization with bubble with non-removable singularity. Since the top bubble in the symplectization always has non-removable singularity, the corresponding second part of the bubbling here is about the asymptotic behavior of a bubble approaching to its non-removable singularities. In particular, it is important to know that along the end, a bubble with non-removable singularity approaches to some closed orbit with an exponential decay rate. In the case that the contact manifold is three dimensional, the desired exponential decay estimate was obtained by Hofer, Wysocki and Zehnder in [HWZ]. It seem that the third part of the bubbling in the contact case, especially, the part concerning the behavior of the connecting tubes along the non-compact **R**-direction was not addressed before. We emphasize that in oder to get the desired compactification without introducing unstable trivial connecting maps, it is crucial to know that the "connecting" tube along the non-compact **R**-direction behaves essentially like the trivial connecting map at C^0 sense. Most analytic part of this paper is aimed to establish the second and the third part of the bubbling process.

Once the above bubbling process is established, the main difficulty to establish the compactification of the moduli space is more conceptual rather than technical. In fact what we need here is a a right definition of stable maps in the contact case, which should incorporate those symmetry splitting mentioned above as well as "hidden" symmetries in each component of the target (See Sec. 3). In particular, according to the consideration in Sec 3, because of these "hidden" symmetries, one should count the **R**-symmetry of each component of the target as many times as the number of the connected components of the domain lying in the component of the target. This will lead to a somewhat "strange" definition of the quivalence of stable maps in the contact case.

In symplectic geometry, historically, the compactness theorem for pseudo-holomorphic maps or connecting (J, H)-maps was first proved by Gromov and Floer [G, F] by adding certain degenerate maps, called cuspidal maps. Later a smaller compactification was found by using stable maps, which plays a important role for the recent development in symplectic geometry (see, for example, [LiT], [FO] and [LT]). Technically, there is not much difficulty to pass from the cuspidal map compactification to stable map one. The key is to carefully keep track all marked points naturally introduced in the bubbling, then to study the deformation of the domain equipped with these marked points in a proper moduli space of curves. In the same vein, the key to get a right compactification in our case is first to understand the two crucial points mentioned in last paragraph, then to keep track carefully all marked points and marked lines in the domain, marked sections in the target naturally appeared in the bubbling and to study the deformation of such a structure. Once the desired compactification is established, our main result about the virtual co-dimension of the boundary will be a consequence of the compactness theorem, the index formula proved in [L3] and a direct dimension counting argument.

As mentioned above, it has been believed that bubbling for pseudo-holomorphic maps in symplectization is a co-dimension one phenomenon. This has been considered as a major difficulty to establish various "simple" and "elementary" constructions, such as Floer homology and G-W invariants, in contact geometry. A very interesting and much more advanced construction were proposed by Eliashberg, Hofer and Givental under the name contact homology or contact Floer homology (see [E]).

On the other hand, the work of this paper and [L1], [L3] suggest a rather different picture on the boundary behavior of the moduli space in contact geometry. This opens the door to construct those "simple" constructions, such as Floer homology and G-W invariants in contact geometry, which have essentially same algebraic structures as the ones in symplectic geometry. It also makes it possible to generalize various important constructions in symplectic geometry. We now briefly mention some of these possible applications, which are outlined in the Sec. 5.

The first application is to define an analogy of Floer homology in contact geometry. To distinguish our construction with the one in [E], which is under the name contact homology or contact Floer homology, we call our construction index homology. The most natural way to do this is to use the closed orbits of the Reeb field to generating a chain group and to count the pseudo-holomorphic maps connecting two closed orbits to define the boundary map. As mentioned above, the co-dimension of the component of broken connecting pseudo-holomorphic maps is one, and bubbling is a co-dimension two phenomenon. Therefore, we are in the exactly the same situation as the usual Floer homology, and the desired index homology can be established as an invariant of the contact structure. Once this is done, one can also construct G-W invariants and use them to define ring structure and the action of the usual homology of the contact manifold on the index homology, which are the analogies in the usual quantum cohomology. To see that the index homology so defined is not always trivial, we introduce Bott-type index homology as a computational tool. Using the Bott-type homology, we can compute the index homology for a contact manifold, which appears as a regular zero locus of some local Hamiltonian function which generates a S^1 Hamiltonian action. It turns out that the index homology of the contact manifold in this case is just the infinite copies of the usual homology of its symplectic quotient indexed by the periods of the closed orbits. Of course the non-vanishing of index homology implies the Weinstein conjecture. Therefore, as a corollary, we proved the Weinstein conjecture in above case.

It is also possible to to use the moduli space differently to define various versions of index homology. In particular, in Sec 5, we will outline how to define an additive quantum homology of a contact manifolds and relative quantum homology of a symplectic manifold with contact type boundary.

There are also some other important constructions that can be generalized. For example, the relative G-W invariant and its gluing formula can be established in general, which was developed by Li-Ruan in [LiR] before with an extra assumption on the existence of some local S^1 Hamiltonian action.

This paper is organized as follows.

In Sec. 2, we will collect and prove some basic facts about the first part of bubbling. Almost all of statements there are well-known due to the work of Hofer and his collaborators. However, for the completeness, we give details of the proof for most of these statements. Besides several technical lemmas in section, the most important thing in section is the introduction of Hofer's energy function, which leads to the important notion of finite energe plane in [H].

In Sec. 3, we formulate the notion of stable maps in the symplectization and the weak-topology of the moduli space of such maps. We then proved the compactness of the moduli space and the statement concerning the co-dimension of its boundary, modulo the statement concerning the exponential decay of a bubble approaching its non-removable singularity and the statement concerning the behavior of the "connecting tube". Both of these statements are proved in Sec. 4.

The last section, Sec 5, is an outline of some possible applications. the detail of these applications will appear in forth coming papers.

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2 Bubbling

Let (M^{2n+1}, ξ) be a contact manifold. This means that ξ is a generic $2n$-dimensional subbundle of TM. A contact form $\lambda = \lambda_\xi$ associated to ξ is a 1-form such that $\lambda \wedge (d\lambda)^n \neq 0$ and $\xi = ker\lambda$. The 2-form $d\lambda$ is non-degenerate when restricted to ξ and has a 1-dimensional kernel at each tangent space of M. We denote by η the line bundle generated by $ker(d\lambda)$. It has a canonic section X_λ defined by requiring that $\lambda(X_\lambda) = 1$. Since $\xi \cap \eta = \{0\}$, we have $TM = \xi \oplus \mathbf{R}\mathbf{X}_\lambda$. Let $\pi : TM \to \xi$ be the projection to the first summand.

- **Symplectization**:

The symplectization of (M^{2n+1}, ξ, λ) is defined as follows.

Let $\widetilde{M}$ be $M \times \mathbf{R}$ equipped with the exact symplectic form $\omega = d(e^r \cdot \lambda)$, where r is the coordinate for the $\mathbf{R}$-factor. Since $d\lambda$ is symplectic along ξ, there exists a $d\lambda$-compatible almost complex structure J defined on ξ. In fact, the set of all such J's is contractible. We extend J to an r-invariant almost complex structure $\tilde{J}$ by requiring:

$$\tilde{J}(\frac{\partial}{\partial r}) = X_\lambda,\ \tilde{J}(X_\lambda) = -\frac{\partial}{\partial r},\ \text{and } \tilde{J} = J$$

along ξ.

- **Equation for $\tilde{J}$-holomorphic curves in $\widetilde{M}$**

Let $\tilde{u} = (u, a) : \Sigma = S^1 \times \mathbf{R} \to \widetilde{M}$ be a $\tilde{J}$-holomorphic map where $u : \Sigma \to M$ and $a : \Sigma \to \mathbf{R}$. Then we have

$$\tilde{J}(\tilde{u}) \circ d\tilde{u} = d\tilde{u} \circ i, \qquad (\star)$$

where i is the standard complex structure on Σ, i.e. $i(\frac{\partial}{\partial s}) = \frac{\partial}{\partial t}$ and $i(\frac{\partial}{\partial t}) = -\frac{\partial}{\partial s}$. Here (s,t) is the cylindrical coordinate of $\mathbf{R} \times S^1$.

Equation $(\star)2$ is equivalent to the following equations:

$$\begin{cases} \pi(u)du + J(u)\pi(u)du \circ i = 0 & (1) \\ (u^*\lambda) \circ i = da & (2) \end{cases}$$

Equation (1) is equivalent to :

$$\pi(u)(\frac{\partial u}{\partial s}) + J(u)\pi(u)(\frac{\partial u}{\partial t}) = 0. \qquad (1')$$

Lemma 2.1 $\Delta a = \frac{\partial^2 a}{\partial s^2} + \frac{\partial^2 a}{\partial t^2} \geq 0$ *if* $\tilde{u}$ *is* $\tilde{J}$*-holomorphic.*

proof: It follows from (2) that

$$\begin{aligned} u^*(d\lambda) &= -d(da \circ i) \\ &= d(-\frac{\partial a}{\partial t}ds + \frac{\partial a}{\partial s}dt) \\ &= (\frac{\partial^2 a}{\partial t^2} + \frac{\partial^2}{\partial s^2})ds \wedge dt. \end{aligned}$$

Now

$$\begin{aligned} u^*(d\lambda) &= d\lambda(\pi(\frac{\partial u}{\partial s}), \pi(\frac{\partial u}{\partial t}))ds \wedge dt \\ &= d\lambda(\pi(\frac{\partial u}{\partial s}), J \cdot \pi(\frac{\partial u}{\partial s}))ds \wedge dt \\ &= g_J(\pi(\frac{\partial u}{\partial s}), \pi(\frac{\partial u}{\partial s}))ds \wedge dt, \end{aligned}$$

where g_J is the Riemannian metric defined on ξ associated with $d\lambda$ and J. Therefore,

$$\Delta a = |\pi(\frac{\partial u}{\partial s})|^2_{g_J} \geq 0.$$

QED

• **Energy**

Let $\phi \in C^\infty(\mathbf{R}, [\frac{1}{2}, 1])$, $\phi' \geq 0$. For any $\tilde{J}$-holomorphic curve $\tilde{u}$, Its ϕ-energy is defined as follows:

$$E_\phi(\tilde{u}) = \int\int_{\mathbf{R}^1 \times S^1} \tilde{u}^* d(\phi\lambda)$$

and its energy

$$E(\tilde{u}) = \sup_\phi E_\phi(\tilde{u}).$$

Let

$$E_\lambda(\tilde{u}) = \int\int_{\mathbf{R}^1 \times S^1} \tilde{u}^* d(\lambda)$$

Note that:

$$
\begin{aligned}
\tilde{u}^*(d(\phi\lambda)) &= \tilde{u}^*(d\phi \wedge \lambda + \phi d\lambda) \\
&= \phi'(u) da \wedge u^*\lambda + \phi(a) u^*(d\lambda) \\
&= \{\phi'(a)\{\frac{\partial a}{\partial s}\lambda(\frac{\partial u}{\partial t}) - \frac{\partial a}{\partial t}\lambda(\frac{\partial u}{\partial s})\} + \phi(a) d\lambda(\frac{\partial u}{\partial s}, \frac{\partial u}{\partial t})\} ds \wedge dt \\
&= \frac{1}{2}\{\phi'(a)\{(\frac{\partial a}{\partial s})^2 + (\frac{\partial a}{\partial t})^2 + \lambda(\frac{\partial u}{\partial s})^2 + \lambda(\frac{\partial u}{\partial t})^2\} \\
&\quad + \phi(a)\{|\pi(\frac{\partial u}{\partial s})|^2 + |\pi(\frac{\partial u}{\partial t})|^2\}\} ds \wedge dt.
\end{aligned}
$$

This implies that $E(\tilde{u}) \geq 0$.

Note: the above local expression $\tilde{u}^*(d(\phi\lambda))$ is valid for any conformal coordinate.

Example

Let $x : S^1 \to M$ be a closed orbit of Reeb field X_λ of period $c = \int_{S^1} \lambda(\dot{x}(t)) dt$. We get a trivial $\tilde{J}$- holomorphic map $\tilde{u}(s,t) = (u, a) = (X(t), c \cdot s)$. Then

$$
\begin{aligned}
E_\phi(\tilde{u}) &= \frac{1}{2}\int_{\mathbf{R}^1 \times S^1} \phi'(a)\{(\frac{\partial a}{\partial s})^2 + \lambda(\frac{\partial u}{\partial t})^2\} ds \wedge dt \\
&= c^2 \int_{-\infty}^{\infty} \phi'(c \cdot s) ds \\
&= c\{\phi(\infty) - \phi(-\infty)\} \\
&= \frac{1}{2} c.
\end{aligned}
$$

• **Bubbling**:

Lemma 2.2 *Let (X, d) be a complete metric space and $\phi : X \to \mathbf{R}^+ = [0, \infty)$ be a continuous function. Given $x \in X$ and $\epsilon > 0$, there exists $x' \in X$ and $\epsilon' > 0$ such that*

(1) $\epsilon' \leq \epsilon$, $\phi(x')\epsilon' \geq \phi(x) \cdot \epsilon$;

(2) $d(x, x') \leq 2\epsilon$;

(3) $2\phi(x') \geq \phi(y)$ for all $y \in X$ such that $d(y, x') \leq \epsilon'$.

The proof is elementary. See [H-V].

Proposition 2.1 *Let $\tilde{u}_n = (u_n, a_n) : \mathbf{R}^1 \times S^1$ (or $\mathbf{C}$)$\to \widetilde{M}$ be a sequence of $\tilde{J}$-holomorphic maps such that (i) there exists a constant $c > 0$ such that $E(\tilde{u}_n) < c$; (ii) for each $\tilde{u}_n$, there exists a $x_n \in \mathbf{R}^1 \times S^1$ such that $|du_n(x_n)| \to \infty$. Then the sequence $\{\tilde{u}_n\}_{n=1}^\infty$ will bubble off at $\{x_n\}_{n=1}^\infty$ a bubble $\tilde{v} : \mathbf{C} \to \widetilde{M}$, which is $\tilde{J}$-holomorphic such that*

(a) $|d\tilde{v}(0)| = 1$;

(b) $|d\tilde{v}(y)| \leq 2$ for any $y \in \mathbf{C}$; and

(c) $E(\tilde{v}) < c$.

Proof:

Apply Lemma 2.2 to the case that $(X, d) = \mathbf{R}^1 \times S^1$, $\phi = |d\tilde{u}_n|$, $x = x_n$, and $\epsilon = \epsilon_n$, where $\{\epsilon_n\}_{n=1}^{\infty}$ is a sequence such that $d_n \cdot \epsilon_n = |d\tilde{u}_n(x_n)| \cdot \epsilon_n \to \infty$. We may assume that

(a) $|d\tilde{u}_n(x_n)| \cdot \epsilon_n \to \infty$;

(b) $2|d\tilde{u}_n(x_n)| > |d\tilde{u}_n(y)|$ for any $y \in D_{\epsilon_n}(x_n)$;

(c) $\tilde{u}_n(x_n) \in M \times \{0\}$ after a translation in $\widetilde{M}$.

Fix $R > 0$, define $\tilde{v}_{n,R} : D_R \to \widetilde{M}$ to be $\tilde{v}_{n,R}(x) = \tilde{u}_n(x_n + \frac{x}{d_n})$.

Note that when n is large enough, $d_n \cdot \epsilon_n > R$. Hence, $x_n + \frac{x}{d_n} \in D_{\epsilon_n}(x_n)$ for $x \in D_R$. This implies that

(i) $|d\tilde{v}_{n,R}(0)| = \frac{1}{d_n}|d\tilde{u}_n(x_n)| = 1$;

(ii) $|d\tilde{v}_{n,R}(x)| = \frac{1}{d_n}|d\tilde{u}_n(x_n + \frac{x}{d_n})| \leq \frac{2d_n}{d_n} = 2$ for any $x \in D_R$;

(iii) $\tilde{v}_{n,R}(0) \in M \times \{0\}$.

Now the standard elliptic estimation implies that $\tilde{v}_{n,R}$ is C^∞-convergent to a $\tilde{J}$-holomorphic map $\tilde{v}_R : D_R \to \widetilde{M}$ after taking a subsequence of $\tilde{v}_{n,R}$. Let $R_n \to \infty$, and taking a diagonal subsequence, $\tilde{v}_{n,R_n}$ is C^∞-convergent to a $\tilde{J}$-holomorphic map $\tilde{v} = \cup_R \tilde{v}_R : \mathbf{C} \to \widetilde{M}$ such that (a) $|d\tilde{v}(0)| = 1$; (b) $|d\tilde{v}(x)| \leq 2, x \in \mathbf{C}$; (c) $E(\tilde{v}) < c$; (d) $\tilde{v}(0) \in M \times \{0\}$.

QED

The following lemma will be used to prove that the bubbling will stop after finite steps.

Lemma 2.3 *Fix $c > 0$. Let V be the collection of all $\tilde{J}$-holomorphic maps $\tilde{v} : \mathbf{C} \to \widetilde{M}$ satisfying the properties (a)-(c) in the previous proposition. Then there exists a constant $\epsilon > 0$ such that for any $\tilde{v} \in V$, $\int_{\mathbf{C}} \tilde{v}^*(d\lambda) > \epsilon$.*

Proof:

If not, there would exist $\tilde{v}_n : \mathbf{C} \to \widetilde{M}$ of $\tilde{J}$-holomorphic maps such that (a) $|d\tilde{v}_n(0)| = 1$; (b) $|d\tilde{v}(x)| \leq 2, x \in \mathbf{C}$; (c) $E(\tilde{v}_n) < c$; and (d) $\tilde{v}_n(0) \in M \times \{0\}$ after $\mathbf{R}$-translation in $\widetilde{M}$; (e) $\lim_{n\to\infty} \int_{\mathbf{C}} \tilde{v}_n^*(d\lambda) = 0$.

Now (a)-(d) implies that $\tilde{v}$ is locally C^∞-convergent to a $\tilde{J}$-holomorphic map $\tilde{v} : \mathbf{C} \to \widetilde{M}$ with same properties of (a)-(d). Now

$$\int_{\mathbf{C}} \tilde{v}^*(d\lambda) = \lim_{R\to\infty} \int_{D_R} \tilde{v}^*(d\lambda) = \lim_{R\to\infty} \lim_{n\to\infty} \int_{D_R} \tilde{v}_n^*(d\lambda) = 0.$$

It follows from Lemma 2.5 that $\tilde{v}$ is a constant map. This contradicts to (a).

QED

Lemma 2.4 *Let $\tilde{u} : \mathbf{R}^1 \times S^1 \to \tilde{M}$ be a $\tilde{J}$-holomorphic map such that $E(\tilde{u}) < \infty$ and $\int_{\mathbf{R}^1 \times S^1} \tilde{u}^*(d\lambda) = 0$. Then either $\tilde{u}$ comes from a closed orbit of X_λ as in Example 1, or $\tilde{u}$ is a constant map.*

Lemma 2.5 *Let $\tilde{u} : \mathbf{C} \to \tilde{M}$ be a $\tilde{J}$- holomorphic map such that $E(\tilde{u}) < \infty$, and $\int_{\mathbf{C}} \tilde{u}^*(d\lambda) = 0$, then $\tilde{u}$ is a constant map.*

Proof of Lemma 2.4

$$\begin{aligned} 0 &= \int_{\mathbf{R}^1 \times S^1} \tilde{u}^*(d\lambda) = \frac{1}{2} \int_{\mathbf{R}^1 \times S^1} \{|\pi(\frac{\partial u}{\partial s})|^2 + |\pi(\frac{\partial u}{\partial t})|^2\} ds \wedge dt \\ &\Longrightarrow \pi(\frac{\partial u}{\partial s}) = \pi(\frac{\partial u}{\partial t}) = 0 \\ &\Longrightarrow u \text{ is tangent to } \mathbf{R}X_\lambda. \end{aligned}$$

This implies that $u = x \circ f$, where $f : \mathbf{R}^1 \times \mathbf{R}^1 \to \mathbf{R}$ and $x = x(t)$ is the solution of $\frac{dx}{dt} = X_\lambda(x(t))$. Here we treat u as a function defined on $\mathbf{R}^1 \times \mathbf{R}^1$ which is periodic in the second variable.

Therefore,
$$\begin{cases} \frac{\partial u}{\partial s} = \dot{x}\frac{\partial f}{\partial s} = \frac{\partial f}{\partial s} X_\lambda(f) \\ \frac{\partial u}{\partial t} = \dot{x}\frac{\partial f}{\partial t} = \frac{\partial f}{\partial t} X_\lambda(f). \end{cases}$$

This implies that $\lambda(\frac{\partial u}{\partial s}) = f_s$ and $\lambda(\frac{\partial u}{\partial t}) = f_t$. Now the equation $\lambda \circ du = -da \circ i$ is equivalent to
$$\begin{cases} \lambda(\frac{\partial u}{\partial s}) = -a_t \\ \lambda(\frac{\partial u}{\partial t}) = a_s. \end{cases}$$

We have
$$\begin{cases} a_s = f_t \\ a_t = -f_s. \end{cases}$$

That is $F = a + fi$ is holomorphic on $\mathbf{C}$. Note that here we treat a as a function on $\mathbf{R} \times \mathbf{R}$ which is periodic on the second variable. Therefor, $F' = \frac{\partial F}{\partial z}$ is also holomorphic.

Now $|dF|^2 = |\frac{\partial F}{\partial z}|^2 = |\frac{\partial a}{\partial z}|^2 + |\frac{\partial f}{\partial z}|^2$. If $|da|$ is bounded, then $|df|$ is also bounded. The holomorphic funtion F' defined on $\mathbf{C}$ has bounded norm, hence, is a constant. This implies that

$$F = c \cdot z + d = (c_1 + c_2 i) \cdot (s + ti) + d_1 + d_2 i = (c_1 s - c_2 t + d_1) + (c_1 t + c_2 s + d_2)i.$$

Hence $f = (c_1 t + c_2 s + d_2)$ and $a(s,t) = c_1 s - c_2 t + d_1$. Sinece a is periodic in t: $a(s, t+1) = a(s,t)$. This implies that $a = c_1 \cdot s + d_1$.

We claim that in this case $x(t)$ is a closed orbit of X_λ, Suppose this is not true, f has to be periodic in t: $f(s, t+1) = f(s,t)$. This implies that $f = c_2 \cdot s + d_1$. Now

But
$$\begin{cases} a_s = f_t = 0 \\ a_t = -f_s = -c_2 \end{cases}$$
implies that $c_1 = c_2 = 0$, and hence F is constant.

Therefore, we may assume that $|da|$, hence $|d\tilde{u}|$ is not bounded. Then the bubbling process described before is applicable to this case and will produce a bubble $\tilde{v} : \mathbf{C} \to \widetilde{M}$ with the properties that (a) $E(\tilde{v}) < \infty$; (b) $\tilde{v}^*(d\lambda) = 0$; (c) $|d\tilde{v}(0)| = 1$; (d) $|d\tilde{v}|$ is bounded. But we will prove in Lemma 5 that (a) and (b) imply that $\tilde{v}$ is a constant map. This contradicts with (c).

QED

Proof of Lemma 2.5

As in Lemma 2.4, we have $u = x \circ f$ and $F = a + fi$ is holomorphic. If $|df|$ and hence $|d\tilde{u}|$ is unbounded, as above, we would have a bubble $\tilde{v} = (v, a)$ with the properties (a)-(d) above. Then (b) implies that $v = x_1 \circ f_1$ and $F_1 = a_1 + f_1 i$ are holomorphic. Now (d) implies that $|da_1|$ and hence $|df_1|$ is bounded. Therefore $F_1' = \frac{\partial}{\partial z} F_1$ has bounded norm. Hence

$$F_1 = (c_1 s - c_2 t + d_1) + (c_1 t + c_2 s + d_2)i$$

and

$$\begin{cases} a_1 = c_1 s - c_2 t + d_1 \\ f_1 = c_1 t + c_2 s + d_2. \end{cases}$$

Now

$$\begin{aligned} \tilde{v}^*(d(\phi \cdot \lambda)) &= \frac{1}{2}\phi'(c_1 s - c_2 t + d_1)\{(\frac{\partial a_1}{\partial s})^2 + (\frac{\partial a_1}{\partial t})^2 + (\frac{\partial f_1}{\partial s})^2 + (\frac{\partial f_1}{\partial t})^2\} \\ &= \phi'(c_1 s - c_2 t + d_1)\{c_1^2 + c_2^2\}. \end{aligned}$$

If c_1 or $c_2 \neq 0$ (say $c_1 > 0$), then

$$\begin{aligned} E_\phi(\tilde{v}^*) &= (c_1^2 + c_2^2) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi'(c_1 s - c_2 t + d_1) ds dt \\ &= \frac{c_1^2 + c_2^2}{c_1} \int_{-\infty}^{\infty} \{\phi(+\infty) - \phi(-\infty)\} dt = +\infty \end{aligned}$$

Hence $c_1 = c_2 \equiv 0$ and $\tilde{v}$ =constant. But this contradicts with (c).

Therefore, $|df|$ and hence $|da|$ is bounded. Hence F' has constant norm. We get $F = cz + d$ again. As above $E(\tilde{u}) < \infty$ implies that $c = 0$.

QED

Proposition 2.2 *Let $\tilde{u} : \mathbf{R}^1 \times S^1$ (or $\mathbf{C}$) $\to \widetilde{M}$ be a $\tilde{J}$-holomorphic map such that $E(\tilde{u}) < \infty$. Then there exists a $c > 0$ such that $|d\tilde{u}(x)| < c$.*

Proof:

$E(\tilde{u}) < c \Longrightarrow \int_{\mathbf{R}^1 \times S^1} \tilde{u}^*(d\lambda) < c' > 0$.

If $|d\tilde{u}|$ is not uniformly bounded, then there exists a sequence $x_n = (s_n, t_n)$ with $s_n \to \pm\infty$ such that $|d\tilde{u}(x_n)| \to \infty$. This will produce a bubble $\tilde{v}$ with the properties (a)-(d) as in Lemma 2.4. Note that (b) $\tilde{v}^*(d\lambda) = 0$ follows from the fact that the s-coordinate of $x_n = (s_n, t_n)$ tends to $\pm\infty$. As in Lemma 2.4, this leads to a contradiction. The proof for the case that $\tilde{u} : \mathbf{C} \to \tilde{M}$ is the same.

QED

Now assume that the contact 1-form λ is generic so that 1 is not an eigen value of the Poincare ăre returning map at any closed orbit of X_λ. This implies that the set of unparameterized closed orbits of X_λ are discrete.

Proposition 2.3 *Let* $\tilde{u} : \mathbf{R}^1 \times S^1 \to \widetilde{M}$ *be a* $\tilde{J}$*-holomorphic map with* $E(\tilde{u}) < \infty$ *and* $\tilde{u} \neq$ *constant map. Then* $\lim_{s \to \infty} \tilde{u}(s,t)$*, when being projected to* M *is either a closed orbit of* X_λ *or a constant map. Assuming the first case happens, then* $\tilde{u}(s,t)$ *is convergent to two closed orbits* $x_\pm$ *asymptotically with a exponential decay rate.*

Proof: The proof for the part concerning the exponential decay of the last statement is given in Sec. 4.

By proposition 2.2, there exists a $C > 0$ such that $|d\tilde{u}| < C$. For any fixed $L > 0$, we define $\tilde{v}_{n,L} = \tilde{u}(s+n,t) : [-L, L] \times S^1 \to \widetilde{M}$. Then $\tilde{v}_{n,L}$ is C^∞-convergent to $\tilde{v}_L$ after taking a subsequence and $\tilde{v} = \tilde{u}(s+n,t)$ is locally C^∞-convergent to $\tilde{v}_\infty : \mathbf{R}^1 \times S^1 \to \widetilde{M}$ such that $E(\tilde{v}_\infty) < \infty$ and $\int_{\mathbf{R}^1 \times S^1} \tilde{v}^*(d\lambda) = 0$. Hence $\tilde{v}_\infty =$ constant map or $\tilde{v}_\infty(s,t) = (x(c \cdot t + d_1), c \cdot s + d_2)$ with $\frac{dx}{dt} = X_\lambda(x)$ and $c = \int_{S^1} x^* \lambda dt$. Note that in the later case, $a(\tilde{v}_n(s,t)) \to \pm\infty$ as $n \to \infty$. We may assume that $c > 0$, and hence $a(s+n,t) \to +\infty$ as $n \to \infty$.

Assume the second case happens. Applying the same argument to the negative end of $\mathbf{R}^1 \times S^1$, we get $\lim_{n \to \infty} p \circ \tilde{u}(s-n,t)|_{[-L,L] \times S^1} = x_-(c_- t + d_-)$ for some closed orbits x_- of X_λ of period c_-, or a constant map. Here p is the projection $\widetilde{M} \to M$. Assume again that it is not the constant map.

Now

$$\int_{\mathbf{R}^1 \times S^1} \tilde{u}^*(d\lambda) = \lim_{n \to \infty} \int_{\{n\} \times S^1} \tilde{u}^*(\lambda) - \int_{\{-n\} \times S^1} \tilde{u}^*(\lambda) = c_+ - c_-.$$

If $\tilde{u}(s+n_i,t)|_{[-L,L] \times S^1}, n_i \to \infty$ is any other convergent sequence, then the limit must also be a closed orbit of period c_+. Let x' be the closed orbit. Under the assumption that λ is generic, there are only finite c-period closed orbits of X_λ. If $x \neq x'$, we can find $(s_i, t_i) \in \mathbf{R}^1 \times S^1$ with $s_i \to +\infty$ and $\tilde{u}(s_i, t_i) \notin$ a small neighborhood of the set of c-closed orbits. Then $\tilde{u}(s+s_i,t)|_{(-L,L) \times S^1}$ is C^∞-convergent to a constant map. This implies that $\int_{\mathbf{R}^1 \times S^1} \tilde{u}^*(d\lambda) = -c_-$ and leads to a contradiction.

Therefore, we conclude that x and x' are the same as unparameterized curves. Then it t is easy to see that as parametrized curve there is also only one limit $\lim_{s \to \pm\infty} p \circ \tilde{u}(s,t) = x_\pm(c_\pm t + d_\pm)$.

In the case of the above limits are closed orbits, $\lim_{s \to \pm\infty} a(s,t) = \pm\infty$. This can be seen easily from the explicit expression of the limit of local convergence of the sequence $\tilde{v}_{n,L}$ introduced at the beginning of the proof.

QED

3 Compactness

Let $M_\pm$ be the two ends of $\tilde{M} = M \times \mathbf{R}$. We consider subset of all finite energy $\tilde{J}$-holomorphic maps whose two ends asymptotically approximate to two closed orbits in $M_\pm$. More precisely, given two parametrized closed orbits $x_\pm : S^1 \to M_\pm \simeq M$, let $\{x\}_\pm$ be the set of all such parametrized closed orbits differ from $x_\pm$ by S^1 actions. Define

$$\widetilde{\mathcal{M}}(x_-, x_+, \tilde{J}) = \{\tilde{u} \,|\, \tilde{u} : \mathbf{R}^1 \times S^1 \to \tilde{M}, \bar{\partial}_{\tilde{J}} = 0, \lim_{s \to \pm\infty} u(s,t) = x'_\pm(t), x'_\pm \in \{x_\pm\}\}.$$

There is an obvious 3-dimensional symmetry group acting on the moduli space. The actions are induced from the $\mathbf{R}$- translations on the target $\tilde{M}$ and $\mathbf{R}^1 \times S^1$-action on the domain $\mathbf{R}^1 \times S^1$. Note that the effect of the two types of actions induced from $\mathbf{R}$-actions on the target and the domain are never identical unless they act on the trivial $\tilde{u} = \tilde{u}(s,t) = (x_\pm(t), s)$.

Let $\mathcal{M}(x_-, x_+, \tilde{J}) = \widetilde{\mathcal{M}}(x_-, x_+; \tilde{J})/\mathbf{R}^2 \times S^1$.

• **Energy**:

Given $\tilde{u} \in \widetilde{\mathcal{M}}(x_-, x_+; \tilde{J})$,

$$\int_{\mathbf{R}^1 \times S^1} \tilde{u}^*(d\lambda) = \int_{\partial(\mathbf{R}^1 \times S^1)} \tilde{u}^*(\lambda) = \int_{S^1} x_+^*(\lambda) - \int_{S^1} x_-^*(\lambda) = c_+ - c_-,$$

where $c_\pm$ are the periods of $x_\pm$.

Lemma 3.1 *Given $u \in \widetilde{\mathcal{M}}(\tilde{x}_-, \tilde{x}_+; \tilde{J})$, then $\int_{\mathbf{R}^1 \times S^1} \tilde{u}^*(d\lambda) \geq 0$ and equality holds if and only if $\tilde{x}_- = \tilde{x}_+$ and $\tilde{u}(s,t) = x_\pm(t)$.*

We will call such $\tilde{u}$ trivial map. Therefore, if $x_- \neq x_+$, $\mathcal{M}(x_-, x_+; \tilde{J})$ does not contain trivial map and the $\mathbf{R}^2$-action is free.

Given $\tilde{u} \in \widetilde{\mathcal{M}}(x_-, x_+; \tilde{J})$,

$$\begin{aligned} E_\phi(\tilde{u}) &= \int_{\mathbf{R}^1 \times S^1} \tilde{u}^*(d(\phi\lambda)) = \int_{\partial(\mathbf{R}^1 \times S^1)} \tilde{u}^*(\phi\lambda) \\ &= \int_{S^1} \phi(a(+\infty)) x_+^*(\lambda) - \int_{S^1} \phi(a(-\infty)) x_-^*(\lambda) \\ &= c_+ - \frac{1}{2} c_- \geq c_+ - \frac{1}{2} c_+ = \frac{1}{2} c_+ > 0 \end{aligned}$$

Compactification of $\mathcal{M}(x_-, x_+; \tilde{J})$:

• Stable $\tilde{J}$-map connecting x_- and x_+:

There are two different ways to define this notion. One is saver but gives less information. We start with this saver one first. The Remark 3.2 in this section will tell us how to modify the definition here to get the more informative one.

Domain $\Sigma = \Sigma_{\tilde{u}}$ of a stable $\tilde{J}$-map $\tilde{u}$ connecting x_- and x_+ can be written as $\Sigma = \cup_i \Sigma_{p_i} \cup_j \Sigma_{b_j}$, $i = 1, \cdots, P$, and $j = 1, \cdots, B$, of the union of domains Σp of its principal components and Σ_b of its bubble components. Each Σ_p or Σ_b is holomorphically equivalent to S^2. As a curve, Σ is semi- stable. This means that the worst singularity of Σ is double point singularity. The components of Σ form a connected tree. There are two particular marked points $-\infty$ on Σ_{p_1} and $+\infty$ on Σ_{p_P}. on each Σ_{p_i}, there are double points $d_{i,-}$ and $d_{i,+}$ such that Σ_{p_i} and $\Sigma_{p_{i+1}}$ are jointed together in Σ at the double point $d = d_{i,+} = d_{i+1,-}$. Therefore, the domain of principal component forms a chain. These joint double points $d_i = d_{i,+} = d_{i+1,-}$ are divided into two classes according to the asymptotic behavior of u when u approaches d_i. We will use I_P to denote the set of those indices i such that u approximates to some closed orbit x_{p_i} when it approaches d_i, while for the other $i \in P \setminus I_P$, $\tilde{u}$ is well defined at $d_{i,+} = d_{i+1,-}$. Similarly, for all other double points of Σ, we will make such a distinction. For each of the double point which is treated as infinity of an end, we will introduce a fix S^1-parameterization at the infinity of the end. We will include this as part of the structure of Σ. This can be done, for example, by identify a small neighborhood U of some double point d with two copies of $R^+ \times S^1$ and using the S^1-parameterization on each of $R^+ \times S^1$ to give the desired S^1-parameterization. For the later application, we mention the following "canonical" way to give the S^1-parameterization for the double point on each of top bubbles in the bubble tree. For each of such bubble, we first add a marked point y, then choose another marked point z along the circle of of the radius 1 centered at y. The ray connecting y and z and started at y gives the required parameterization at infinity. We remark that it is only the parameterization itself is included in the structure of Σ, not the other things used to define it. Therefore the dimension of the symmetry group of a top bubble is three.

Note that our definition of the domain of a stble map is similar to the one used in the usual Gromov-Floer theory. However if we restricted to the compactification of $\mathcal{M}(x_-, x_+; \tilde{J})$, then the domains of its elements subject to further restrictions. Although it does not effect constructions in this paper and the subsequent forth coming papers in any essential way, these furth restrictions simplify the possible intersection pattern of domains and make the situation here is different from the corresponding case in the Gromov-Floer theory. We refer the readers to Remark 3.2 of this section on this.

The target U of $\tilde{u}$ is a union $U = \cup_{i \in I_P} \tilde{M}_{p_i}$ with each $\tilde{M}_{p_i} \simeq \tilde{M}$. Note that here we have somewhat abused the notation as it may happen that on each $\tilde{M}_{p_i}$, there may exist more than one $\tilde{u}_{p_i}$'s. Each $\tilde{M}$ has two ends $\tilde{M}_{p_i,\pm}$ and we identify $\tilde{M}_{p_i,+}$ with $\tilde{M}_{P_{i+1},-}$. On each $M_{p_i,\pm}$, there is a particular closed orbit $x_{p_i,\pm}$ associated to each index p_i, $i \in I_P$ and possibly some other closed orbits $x_{p_i,b_{j,l}}$, $i = 1, \cdots, P$. Here $b_{j,l}$ are indices of the double points on bubble component Σ_{p_i,b_j}. Here we have relabeled bubble component Σ_{b_k} before as Σ_{p_i,b_j}, where p_i is principal component on which the bubble Σ_{b_k} lies.

Note that x_- lies on the negative end of the first $\tilde{M}_{p_i}$'s and x_+ lies on the positive end of the last $\tilde{M}_{p_i}$'s, $p_i \in I_P$.

The stable map $\tilde{u} = \cup_{i=1}^{P} \tilde{u}_{p_i} \cup_{j=1}^{B} \tilde{u}_{p_i,b_j}$ such that

(i) $\tilde{u}_{p_i} : \Sigma_{p_i} - \{\text{double points}\} \to \tilde{M}_{\phi(p_i)}$ and $\tilde{u}_{b_j} : \Sigma_{p_i,b_j} \setminus \{\text{double points}\} \to \tilde{M}_{\phi(p_i)}$ are $\tilde{J}$- holomorphic. Here $\phi(p_i)$ is a function from the set of indices $\{1, \cdots, P\}$ to I_P, which is the identity map when being restricted to I_P.

(ii) Along each end near the double point $d_i \in \Sigma_{p_i}$, $\tilde{u}_{p_i}$ is convergent exponentially to some parametrized periodic orbit x_{p_i}, if $i \in I_P$. Otherwise, $\tilde{u}_{p_i}$ is well-defined at d_i and $\tilde{u}$ has an ordinary double point at d_i. Similarly at each double point on bubble components or double point on principal components other than these d_i's, $\tilde{u}$ either asymptotically approximates to a closed orbit x or it extends smoothly across these double points. Note that in the case that $\tilde{u}$ asymptotically approximates to a parameterization closed orbit along some double point, the S^1-parameterization (covering) of the closed orbit is given by the S^1-parameterization of the end.

(iii) On each $\tilde{M}_{p_i}$, $i \in I_P$, there is an $\mathbf{R}^1$- action of r-translation. We require that the isotropy subgroup of the components of $\tilde{u}$ in $\tilde{M}_{P_i}$ is not the entire $\mathbf{R}^1$. This implies that each $\tilde{M}_{p_i}$ contains at least one bubble components if the principle component of $\tilde{u}$ in $\tilde{M}_{p_i}$ is a trivial component. Here a trivial principal component of $\tilde{u}$ in $\tilde{M}_{p_i}$ is the $\tilde{J}$-holomorphic map $\tilde{u}_{p_i} : \mathbf{R}^1 \times S^1 \to \tilde{M}_{p_i}$ such that $\tilde{u}_{p_i}(s,t) = (x(ct), cs + d)$ for some periodic orbit x in M. Clearly the isotropy group of such an $\tilde{u}_{p_i}$ is $\mathbf{R}^1$ itself.

(iv) Each constant bubble component is stable in the sense that it contains at least three double points.

(v) $\tilde{u}$ connects x_- and x_+, meaning that it connects some $x'_- \in \{x_-\}$ and $x'_+ \in \{x_+\}$.

Note that similar to the stable maps used in the usual Floer homology, there are two kinds of trivial components, and the trivial principal components play similar role as closed orbits of a Hamiltonian system regarded as trivial principal components of a stable (J, H)-map in Floer homology. The reason to rule out this kind of components can be seen as follows.

Example

Let $\tilde{u} : S^1 \times \mathbf{R}^1 \to \tilde{M}$ be a $\tilde{J}$- holomorphic map connecting two closed orbits x_- and x_+. Assume that $\tilde{u}$ is not trivial. Hence the effect of the $\mathbf{R}$-actions on $\tilde{u}$ induced by the $\mathbf{R}$-actions on $\tilde{M}$ is different from those induced by the $\mathbf{R}$-actions on the domain $S^1 \times \mathbf{R}^1$. Define $\tilde{u}_n(s,t) = \tilde{u}(s+n,t)$. Then $\{\tilde{u}_n\}_{n=0}^{\infty}$ is locally C^∞-convergent to $\tilde{u}_\infty = \tilde{u}_{\infty,0} \cup \tilde{u}_{\infty,1}$ where $\tilde{u}_{\infty,0} = \tilde{u}$ and $\tilde{u}_{\infty,1} : \mathbf{R}^1 \times S^1 \to \tilde{M}_1$ with $\tilde{u}_{\infty,1}(s,t) = (x_+(ct), cs + d)$. Iterating this process, we can produce any number of trivial principal components as limit.

Note that this example also indicates that even though the energy $E(\tilde{u})$ is a constant for any $\tilde{u} \in \widetilde{\mathcal{M}}(x_-, x_+; J)$, it is not preserved when passing to the limit. We will see that for the element $\tilde{u}$ in the moduli space $\tilde{u} \in \widetilde{\mathcal{M}}(x_-, x_+; J)$ of stable connecting maps, the energy is uniformly bounded. However, without the assumption of stability, there is no such a bound as above example shows. On the other hand, the quantity $\int_{\mathbf{R}^1 \times S^1} \tilde{u}^* d\lambda$ is obviously preserved under the limit process.

• **Compactness**

Let

$$\overline{\mathcal{M}}(x_-, x_+; J) = \{[\tilde{u}] \,|\, \tilde{u} \text{ is a stable } \tilde{J}\text{-map connecting } x_- \text{ and } x_+;\ E(\tilde{u}) \text{ is finite.}\}.$$

Here $[\tilde{u}]$ is the equivalent class of $\tilde{u}$.

The definition here needs some explanation. In the usual quantum homology and Floer homology in sympletic geometry, to form the moduli space $\mathcal{M}(x_-, x_+; J)$ and its compactification, one needs to fix a relative homotopy class, which is represented by the elements in these moduli spaces. Therefore, there are options here. One is to follow the usual definition, which is saver but less informative. We will leave the rutin formulation of this saver definition to our reader. On the other hand, the Remark 3.2 together with the next two lemmas imply that. One still can prove the compactness without restricting to a particular relative homotopy class.

Definition 3.1 *Two stable $\tilde{J}$-map $\tilde{u}_1$ and $\tilde{u}_2$ connecting x_-, x_+ are said to be equivalent if there exists an equivalence $\phi : \Sigma_1 \to \Sigma_2$ of their domains and an equivalence $\psi : \tilde{U}_1 \to \tilde{U}_2$ of the liftings of their targets such that $u_1 = \psi^{-1} \circ u_2 \circ \phi$, where ϕ is a homomorphism of Σ_1 and Σ_2 such that it is bi-holomorphic along each of their components and preserves the variable $t \in S^1$ along the chain of principal components and preserves the S^1-parameterizations at infinity along those ends of bubble components or principal components approaching to some closed orbits, and ψ is induced from $\mathbf{R}$-translations on each component of the target U, in the sense explained in the following. Here $\tilde{U}_1$ and $\tilde{U}_2$ are certain finite liftings determined by the connected components of the bubble tree in each of the components of U's.*

Note that the components of the domain Σ of a stable map $\tilde{u}$ forms a connected tree. If we fix a component $\tilde{M}_i = \tilde{M}$ of the target U, and collect those components of the domain in the bubble tree above whose images stay in $\tilde{M}_i$, the components may not be connected anymore. We will associated to each of a connected components $\Sigma_{i,j}, j = 1, \cdots, J_i$, of the domain in $\tilde{M}_i$, a $\tilde{M}_{i,j} = \tilde{M}$. We collect all of these $\tilde{M}_{i,j}$ together with same ends as before as the lifting of U mentioned above. Then the ψ defined above is just induced by the $\mathbf{R}$-translations on each component of $\tilde{U}$'s. In particular, it follows from this definition that on each component of the target of a stable map, there are as many dimensions of $\mathbf{R}^1$-actions as the number of the connected components of the domain in the component of the target.

There is a special case that the above definition on connected component of the domain in a component of the target is not applicable. According to above definition, if a component of the target contains a trivial connecting map, the domain of this map clearly is an isolated component in the domain of the orignal map inside the component of the target. For the obvious reason that one can not assign an extra $\mathbf{R}^1$-symmetry of the target associated the the trivial map. However, as we will prove in this section that on the component of the target, there exists at least one non-trivial connected component of the domain.

We simply define a connected component in this case as the union of the non-trivial one with the domain of the trivial map. In the case that there are several non-trivial connected components together with several trivial maps in one component of the target, we consider all possible combinations of them and consider this as part of the data in the definition of stable map. Another way to deal with this particular case is to only count the $\mathbf{R}^1$-symmetry of the domain for each of such stable component.

With this interpretation, we note that with respect to the symmetry group so defined, the isotropy group of a stable map is always finite, which is important for defining the virtual moduli cycles in [L1] and will be proved there.

Theorem 3.1 *$\overline{\mathcal{M}}(x_-, x_+; J)$ is compact and Hausdorff with respect to the C^∞-weak topology, which is a compactification of $\mathcal{M}(x_-, x_+; J)$.*

Proof:

The Hausdorffness follows from the stability of elements in $\bar{\mathcal{M}}(x_-, x_+; J)$. The proof of the corresponding theorem in [LT] can be easily adapted here. We refer our readers to the proof there.

Lemma 3.2 *There exists a constant $N = N(x_-, x_+)$ such that for any $\tilde{u} \in [\tilde{u}] \in \overline{\mathcal{M}}(x_-, x_+; J)$, the number of components of u is less than N.*

Proof:

By using local convergence, one can easily show that there exists a fixed $\epsilon > 0$, such that for any non-trivial bubble $\tilde{u}_b$, $E_\lambda(\tilde{u}_b) > \epsilon$. Same conclusion holds for non-trivial principal component $\tilde{u}_p : S^1 \times \mathbf{R}^1 \setminus \{\mathrm{d}ouble\, points\} \to \tilde{M}$. Note that here we used $x_- \neq x_+$.

Since

$$\begin{aligned} E_\lambda(\tilde{u}) &= \Sigma_{b,p} E_\lambda(\tilde{u}_b) + E_\lambda(\tilde{u}_p) \\ &= \int_{S^1} x_+^* \lambda - \int_{S^1} x_-^* \lambda = c_+ - c_- \end{aligned}$$

is fixed, we only need to prove that the number of trivial principal components and trivial bubble components is uniformly bounded.

Now for each trivial principal component, there exists some non-trivial bubble components lying on the same target component. Therefore, the number of such components is less than or equal to the number of non-trivial bubble components, which is bounded. Finally, it is easy to see inductively that the number of trivial bubble components can be uniformly bounded by the number of non-trivial bubbles and principal components.

Lemma 3.3 *Given $\tilde{u} \in [\tilde{u}] \in \overline{\mathcal{M}}(x_-, x_+; J)$, if x is a closed orbit such that it is an intermediate end of some component of $\tilde{u}$. Then $\int x^* \lambda < \int x_+^* \lambda = c_+$. Therefore, there are only finite such closed orbits.*

Proof:

$x_+ \subset M_{p,+}$ is the only closed orbit lying on the positive end of $\tilde{M}_p$, where $\tilde{M}_p$ is the most right (positive) components of the target of $\tilde{u}$. Then

$$\int x_+^* \lambda - \Sigma_i \int x_{-,i}^* \lambda = \int_{u|\tilde{M}_p} d\lambda > 0,$$

where $x_{-,i}$ is one of the closed orbits on $\tilde{M}_{p,-}$ appeared as a non-trivial end of $\tilde{u}$. The conclusion follows by induction.

To see that there are only finite such intermediate x, we use Remark 3.2. It follows from the remark there that $\int x^* \lambda$ is bounded below by $\epsilon > 0$ of the lower bound of the E_λ-energy of non-trival bubbles.

QED

It follows from this argument that the number of double points of a component of a stable map appeared in the compactification is also bounded.

Because of these lemma, the proof of the theorem essentially can be reduced to the case that $[\tilde{u}_i]_{i=1}^{\infty}$ has only one component and we need to show that such a sequence has a weak-limit $[\tilde{u}_\infty] \in \widetilde{\mathcal{M}}(x_-, x_+; J)$.

Note that the above two lemmas together implies that the energy of $E(\tilde{u})$ is uniformly bounded for any $\tilde{u} \in \widetilde{\mathcal{M}}(x_-, x_+; J)$.

• **Stabilization of the target and its local deformation**

The target of $\tilde{u}$ of a stable map is a union $U = \cup_{i \in I_P} \tilde{M}_{p_i}$. Each $\tilde{M}_{p_i} \simeq \tilde{M} = M \times \mathbf{R}$ with a $\mathbf{R}^1$-symmetry coming from the r-translations along the second factor $\mathbf{R}$. To stabilize U, we add a marked point z_i on the second factor of $\tilde{M}_{p_i}$ to remove theses symmetries. Given $(\tilde{M}_i, z_i) = M \times (\mathbf{R}, z_i)$, $i = 1, 2$ and $\tau \in \mathbf{R}^+$ of a deformation (gluing) parameter, we can form $U_\tau = (\tilde{M}_1, z_1) \#_\tau (\tilde{M}_2, z_2)$ of the local deformation of $U = (\tilde{M}_1, z_1) \cup (\tilde{M}_2, z_2)$ with respect to the parameter τ by the obvious gluing construction along the second factor of $\tilde{M}_1$ and $\tilde{M}_2$, namely, cutting off $r_1 > \frac{1}{\tau}$ of $\tilde{M}_1$ and $r_2 < -\frac{1}{\tau}$ and gluing back the remaining parts. Then $U_\tau \simeq M \times \mathbf{R}$ with two marked points z_1 and z_2 on $\mathbf{R}$. Similarly, if $U = \cup_{i \in I_P} \tilde{M}_{0_i}$ and $\tau = (\tau_1, \cdots, \tau_{\gamma-1}) \in (\mathbf{R}^+)^{\gamma-1}$, $\gamma = \#(I_P)$, with γ marked points $z_i \in \tilde{M}_{p_i}$, $i = 1, \cdots, \gamma$, we can form $U_\tau = \tilde{M}_{p_1} \#_{\tau_1} \tilde{M}_{p_2} \# \cdots \#_{\tau_{\gamma-1}} \tilde{M}_{p_\gamma}$ with marked points $z_1, \cdots, z_\gamma$ on it. Another way to think this is to treat the marked point z_i as a marked section $M \times \{z_i\}$ in $\widetilde{M}_i$.

A cylinder $\tilde{M} = M \times (\mathbf{R}; -\infty, +\infty, z_1, \cdots, z_n)$ with two end points $-\infty$ and $+\infty$ and n distinct marked points $z_1, \cdots, z_n$ is said to be stable of type M if $n \geq 1$. let $\mathcal{M}_n(M)$ be the collection of all such stable cylinders of n marked points of type M. Then it has an obvious compactification

$$\bar{\mathcal{M}}_n(M) = \mathcal{M}_n(M) \coprod_{\substack{l+m=n \\ l,m \geq 1}} \mathcal{M}_l(M) \times \mathcal{M}_m(M)).$$

The topology of $\bar{\mathcal{M}}_n(M)$ near the boundary points is described by the local deformation (gluing) above.

• **Weak Convergence**

• Stabilization of a semi-stable curve and its local deformation:

Domain Σ of a stable map $\tilde{u}$ is only a semi-stable curve. Therefore, there may exist some non-trivial bubble components or principal components whose domains contain only one or two double points. We can stabilize these unstable components by adding minimal number of marked points $\underline{y} = (y_1, \cdots, y_m)$ to get a stable curve $(\Sigma, \underline{y})$. In particular, for each top (hence, unstable) bubble, the symmetry group is three dimensional because of extra structure of the S^1-parametrization at infinity along its end. To stabilize such a component we introduce an arbitary marked point y_1 first. Then the S^1-parametrization at infinity together with the marked point determine a marked ray connecting y_1 to $\theta = 0$ at S^1 at infinity an obvious way. We add the second marked point y_2 on the marked ray with distance of 1 to y_1 to get the desired stablization. Let $(\Sigma_\alpha, \underline{y})$ the local deformation of $(\Sigma, \underline{y})$ in the moduli space of stable curves, where α is the collection of deformation parameters associated with double points of Σ. Note that the moduli space of stable maps used here is not the ususal Degline-Mumford compactification but an obvious modification of the moduli space of stable (J, H)-maps used in [LT]. Here for each ordinary double point of $\tilde{u}$, we associate it with a complex gluing parameter and for each double point corresponding to an end approaching to closed orbit, we associate a positive real gluing parameter.

To see this more concretely, we consider the following example.

Example

Consider the semi-stable curve $\Sigma = P \cup_{d_1 = d_2} B$ with principal component $P = \mathbf{R} \times S^1$ and bubble B joint at the double point d. Assume that the double point corresponds to the two end on P and B. Not that on P there are other two marked points corresponding to $-\infty$ and $+\infty$ in $\mathbf{R} \times S^1$. The moduli space of such semi-stable map is 1-demensional due to the choices of S^1-parametrization at d. To stable such a Σ, we only need to stabilize B, which is described above. Associated to the double point, there is a one dimensional local deformation of Σ with respect to a gluing parameter $\alpha \in \mathbf{R}^+$. By letting Σ vary in above 1-dimension moduli space, the deformation gives the elements in the moduli space $\mathcal{M}_{0,4}$, which form a neighbourhood the the 1-dimensional moduli space. Therefore, the above 1-dimensional moduli space can be thought as part of the boundary of $\mathcal{M}_{0,4}$. Of course, this not the usual Degline-Mumford compactification of $\mathcal{M}_{0,4}$. On the other hand, give any sequence $\Sigma_i \in \mathcal{M}_{0,4}$ with $\Sigma_i = (\mathbf{R} \times S^1; -\infty, +\infty, y_i, z_i$ with $y_i \to z_i$ as in the bubbling below, the process there will give a limit of the sequence in the above 1-dimensional moduli space.

• **Defintion of Weak Convergence**:

Given $[\tilde{u}_i]_{i=1}^{\infty} \in \overline{\mathcal{M}}(x_-, x_+; \tilde{J})$, we say that $[\tilde{u}]$ is weakly C^∞-convergent to $[\tilde{u}_\infty] \in \overline{\mathcal{M}}(x_-, x_+; \tilde{J})$ if there exist $\tilde{u}_i \in [\tilde{u}_i]$ and $\tilde{u}_\infty \in [\tilde{u}_\infty]$ such that

(i) After adding some marked points $\underline{y}_i$ to Σ_i, $(\Sigma_i, \underline{y}_i)$ is convergent to the minimal stabilization of $(\Sigma_\infty, \underline{y}_\infty)$ in the moduli space of stable curves, where Σ_i and Σ_∞ are the domains of $\tilde{u}_i$ and $\tilde{u}_\infty$. Note that the number of marked points $\underline{y}_i$'s is same as the number of marked points $\underline{y}_\infty$. Therefore, when i is large enough, there exists an α_i such that $(\Sigma_i, \underline{y}_i)$ is equivalent to $(\Sigma_{\infty,\alpha_i}, \underline{y}_\infty)$ of the deformation of $(\Sigma_\infty, \underline{y}_\infty)$ with respect to the gluing parameter α_i.

Let $\phi_i : (\Sigma_{\infty,\alpha_i}, \underline{y}_\infty) \to (\Sigma_i, \underline{y}_i)$ be the equivalence map.

(ii)Let $U_i = \cup_{j\in I_{P_i}} \tilde{M}_{p_{i,j}}$ and $U_\infty = \cup_{j\in I_{P_\infty}} \tilde{M}_{p_\infty}$, be the targets of $\tilde{u}_i \in [\tilde{u}_i]_{i=1}^\infty$ and $\tilde{u}_\infty \in [\tilde{u}_\infty]$. We stabilize U_∞ by adding minimal number of marked points $\underline{z}_\infty$ and require that after adding same number of makred points $\underline{z}_i$ to U_i, $(U_i, \underline{z}_i)$ is convergent to $(U_\infty, \underline{z}_\infty)$ in the space of $\bar{\mathcal{M}}_n(M)$. Here n is the number of marked points of $\underline{z}_\infty$. Therefore, there exists τ_i such that $(U_i, \underline{z}_i)$ is equivalent to $((U_\infty)_{\tau_i}, \underline{z}_\infty)$.

Let $\psi_i : (U_i, \underline{z}_i) \to ((U_\infty)_{\tau_i}, \underline{z}_\infty)$ be the equivalence map. Note that for u_i closed to u_∞, the gluing parameter α_i of the domain $(\Sigma_i, \underline{y}_i) \equiv ((\Sigma_\infty)_{\alpha_i}, \underline{y}_\infty)$ is not compleletely independent of the gluing paprameter τ_i of the targe $((U_\infty)_{\tau_i}, \underline{z}_\infty)$ since along these ends where $[u_\infty]$ approaches closed orbit $\alpha_i = 0 \Longleftrightarrow \tau_i = 0$. However, when $\alpha_i \neq 0$, hence $\tau_i \neq 0$, they are essentially independent each other.

(iii) Given a compact set $K \subset \Sigma_\infty \setminus \{\text{double points}\}$, the compact image $\tilde{u}_\infty(K) \subset U_\infty \setminus \{\text{end of } U_\infty\}$. Hence for i large enough, $\tilde{u}_\infty(K) \subset (U_\infty)_{\tau_i} \equiv U_i$. Therefore, $\psi_i^{-1} \circ u_\infty$ is well-defined on K and it maps K into U_i. On the other hand, for large i, $K \subset (\Sigma_\infty)_{\alpha_i}$ and $\phi_i(K) \subset \Sigma_i$, and $\tilde{u}_i \circ \phi_i : K \to U_i$. We require that (a) $\tilde{u}_i \circ \phi_i|_K$ is C^0-close to $(\psi_i^{-1} \circ \tilde{u}_\infty)|_K$ when i is large enough, hence, $\psi_i \circ u_i \circ \phi_i|_K : K \to U_\infty$ is well-defined. (b) for any compact subset $K \subset \Sigma_\infty \setminus \{\text{double points}\}$, $\psi_i \circ \tilde{u}_i \circ \phi_i|_K$ is C^∞-convergent to $\tilde{u}_\infty|_K$.

Note that $E_\lambda(\tilde{u}_i) = E_\lambda(\tilde{u}_\infty) = c_+ - c_-$ is fixed. This together with the two statements of Sec. 4. imply that the projections of the images of $\tilde{u}_i$ to the contact manifold M is C^0-close to the projection of the image $\tilde{u}_\infty$, and that near a closed orbit x with λ-period c as an asymptotic end of $\tilde{u}_\infty$, along the non-compact $\mathbf{R}$-direction of $\tilde{M}$, $\tilde{u}_i$ is essential same as the function $c \cdot s$, when i is large enough.

We start with a detailed description on the case that the sequence $[\tilde{u}_i]$ only develops one bubble, as it already exhibits all of the main points of the general case. The following lemma plays an important role both in the proof of this theorem and in the later formal dimension counting of the boundary of the modui space of $\tilde{J}$- holomorphic maps.

Lemma 3.4 *If $\{[\tilde{u}_i]\}_{i=0}^\infty \in \mathcal{M}(x_-, x_+; \tilde{J})$ develops only one bubble at its limit $[u_\infty]$, then the target U_∞ contains at least two elements and the image of bubble is not in the right most component. This implies that the domain Σ_∞ of u_∞ contains at least three components.*

Proof:

Let $\tilde{u}_i \in [\tilde{u}_i]$, $\tilde{u}_i = (u_i, a_i) : \Sigma_i = \mathbf{R}^1 \times S^1 \to \widetilde{M}$, where $u_i : \mathbf{R}^1 \times S^1 \to M$ and $a_i : \mathbf{R}^1 \times S^1 \to \mathbf{R}$. By assumption, there exists bubble point $y_i \in \Sigma_i$ such that $|d\tilde{u}_i(y_i)| \to \infty$. First assume that y_i stays in a compact set of $\Sigma_i = \mathbf{R}^1 \times S^1$, hence $y_i \to y_\infty \in \mathbf{R}^1 \times S^1$ as $i \to \infty$ after taking a subsequence. We claim that $|a_i(y_i)|$ is not bounded and $a_i(y_i)$ tends to $-\infty$. Otherwise, assume that $|a_i(y_i)| < C$. Then $a_i(y_i) \to a_{\infty,y} \in \mathbf{R}$. Now the domain of Σ_i has two marked points y_i and w_i, where w_i is a point on the circle centered at y_i of radius $\frac{1}{|d\tilde{u}_i(y_i)|}$. Note that here we can make an arbitary choice for w_i on the circle. See the remark after on how to make "correct" choice. These two marked points y_i, w_i together with $-\infty, +\infty$ on $S^2 = \Sigma_i \cup \{-\infty, +\infty\}$ have moduli, and we can identify the domain $(\Sigma_i, -\infty, +\infty, y_i, w_i)$ with $(\Sigma_i, -\infty, \infty, d)\#_{\alpha_i}(S^2, 0, 1, d')$. Here $(\Sigma_i; -\infty, \infty, d)\#_{\alpha_i}(S^2; 0, 1, d')$ is obtained from $(\Sigma_i; -\infty, \infty, d)\bigvee_{d=d'}(S^2; 0, 1, d')$ by gluing at d with some deformation parameter $\alpha_i \in \mathbf{R}^+$ with $\alpha_i \to 0$ as $i \to \infty$, and $(\Sigma_i; -\infty, \infty, d)\bigvee_{d=d'}(S^2; 0, 1, d')$ is one of the elements in the 1-dimensional moduli space mentioned in the previous example. In particular, along the end d, there is a S^1-parametrization. Intuitively, what we did here is to

conformally enlarg a small disc of Σ_i near y_i, bringing y_i, w_i into standard points $0, 1$ in standard disc.

Now $(\Sigma_i\#_{\alpha_i}S^2; -\infty, \infty, 0, 1)$ has four marked points $-\infty, \infty, 0, 1$, and

$$(\Sigma_i\#_{\alpha_i}S^2; -\infty, \infty, 0, 1) \simeq (\Sigma_i; -\infty, \infty, y_i, w_i).$$

Let $\phi_i : (\Sigma_i\#_{\alpha_i}S^2; -\infty, \infty, 0, 1) \to (\Sigma_i; -\infty, \infty, y_i, w_i)$ be the identification map. Let D_R be the half shpere glued with a finite cylinder $S^1 \times [0; R]$ along its boundary. We still use D_R to denote its obvious conformal image in $(S^2; 0, 1, d') \subset (\Sigma_i; -\infty, \infty, d)\bigvee_{d=d'}(S^2; 0, 1, d')$ centered at 0, and $D_{R,i}$ the corresponding image in $\Sigma_i\#_{\alpha_i}S^2$ when i is large enough.

Define $\widetilde{V}_{i,R} = (\tilde{u}_i \circ \phi_i)|_{D_{R,i}}$. Then as we did before for bubbling, $\tilde{v}_{i,R} \to \tilde{v}_{\infty,R} : D_R \to \widetilde{M}$ and $\tilde{v}_i = \tilde{u}_i \circ \phi_i$ is locally C^∞-convergent to $\tilde{v}_\infty = \cup_R \tilde{v}_{\infty,R} : D_\infty = D^2 \cup (\mathbf{R}^+ \times S^1) \to \widetilde{M}$. That is $\{\tilde{u}_i\}$ produce a bubble at y_i. The domain of $\tilde{v}_\infty$ is the complex plane but thought as half sphere with a half infinite cylinder attached. Since $\tilde{v}_\infty$ is $\tilde{J}$-holomorphic and $E(\tilde{v}_\infty) < \infty$, $|D\tilde{v}_\infty|$ is uniformly bounded. As before, $\lim_{s\to+\infty} v_\infty(s,t) = x(t)$ of some periodic orbit along its cylindrical end. Now fix $\epsilon > 0$, and consider $\tilde{u}_{i,\epsilon} = \tilde{u}_i|_{\Sigma_i \setminus D_\epsilon(y_i)}$. By our assumption that there is only one bubble we conclude that for any fixed $\epsilon > 0$, $|d\tilde{u}_{i,\epsilon}| < C_\epsilon$ for any i. We may assume that $\lim_i \tilde{u}_i(0,0)$ exists at the begining and $y_i \neq (0,0)$. Then the same argument as before implies that $\tilde{u}_{i,\epsilon}$ is C^∞-convergent to $\tilde{u}_{\infty,\epsilon} : \mathbf{R}^1 \times S^1 \setminus D_\epsilon(y_\infty) \to \widetilde{M}$. Here we used that $|y_i|$ is bounded and hence $y_i \to y_\infty \in \mathbf{R}^1 \times S^1$. By letting $\epsilon \to 0$, we get $\tilde{u}_i|_{\Sigma_i \setminus \{y_i\}}$ is locally C^∞-convergent to $\tilde{u}_\infty|_{\mathbf{R}^1 \times S^1 \setminus \{y_\infty\}}$. Identifying $D_\epsilon(y_\infty) - \{y_\infty\}$ ($\subset \mathbf{R}^1 \times S^1 - \{y_i\}$) with $\mathbf{R}^+ \times S^1$, then $\lim_{s\to\infty} u_\infty(s,t) = x'(t)$ of a closed orbit.

Let $\tilde{v}_\infty = (v_\infty, b_\infty)$. Then $\lim_{s\to\infty} b_\infty(s,t) = +\infty$. Otherwise, since $b_\infty(s,t) \sim cs + d$ with $c = \int_{S^1} x^*\lambda \neq 0$, we have $\lim_{s\to\infty} b_\infty(s,t) \to -\infty$. But since $\Delta b_\infty \geq 0$, this contradicts to the maximal principle for sub-harmonic functions.

Therefore, $b_\infty(s,t) \sim cs + d$ with $c = \int_{S^1} x^*\lambda > 0$. The induced orientation of $\tilde{v}_\infty$ on x is the same as the one given by λ. By our assumption that there is only one bubble, if we set $\tilde{u}_\infty = (u_\infty, a_\infty)$, then $\lim_{s\to\infty} a_\infty(s,t) = +\infty$, $\lim_{s\to\infty} u_\infty(s,t) = x'(t) = x(t)$. Here, $(s,t) \in \mathbf{R}^+ \times S^1 = D_\epsilon(y_\infty) \setminus \{y_\infty\}$.

This implies that the induced orientation on $x'(t) = x(t)$ form $\tilde{u}_\infty$ is also the same as the one given by λ. However, $\tilde{u}_\infty \cup \tilde{v}_\infty$ is the weak limit of $\tilde{u}_i$ and $x(t) = x'(t)$ is the limit of some corresponding curves x_i in $\tilde{u}_i$. Clearly, the induced orientations of x_i obtained from the two sides of $\tilde{u}_i$ are opposite to each other. This is a contradiction.

We remark that one can also get an alternative proof of above statement by using gluing in [LT] and maximal principle instead of using this orientation consideration.

This proves that $|a_i(y_i)|$ is not bounded under the assumption that $|y_i|$ is bounded. In the case that $y_i \to \pm\infty$, $(\mathbf{R}^1 \times S^1; y_i, 0, -\infty, \infty)$ tends to a boundary point of moduli space $\bar{\mathcal{M}}_{0,4}$. If, say, $y_i \to \infty$, let $(S^2; -\infty, y_\infty, d) \bigvee_{d=d'} (S^2, d', 0, +\infty)$ be the limit curve. Then $(\mathbf{R}^1 \times S^1; -\infty, y_i, 0, \infty) \simeq (S^2 \#_{\alpha_i} S^2; -\infty, y_\infty, 0, \infty)$ for some $\alpha_i \in \mathbf{C}^*$. Now in $S^2 \#_{\alpha_i} S^2$, y_∞ plays the same role y_i in Σ_i but it stays away from the two ends. The above argument is still applicable except that at the limit, the domain has one more splitting.

Therefore, $a_i(y_i) \to \pm\infty$. If $a_i(y_i) \to +\infty$, after shifting by $-a_i(y_i)$ to the target of $\tilde{u}_i$ and define $\tilde{w}_i = (u_i, a_i - a_i(y_i))$, the above argument is still applicable to $\tilde{w}_i$, and we get bubble at y_∞, still denoted by $\tilde{v}_\infty = (v_\infty, b_\infty)$. In particular, $\lim_{s\to\infty} b_\infty(s,t) = +\infty$. Therefore, we get a bubble as before but with target $\widetilde{M}'$ lying on the right of $\widetilde{M}$ with the end of the bubble approaching a closed orbit lying on the right end of $\widetilde{M}'$. As before, the orientation consideration and maximum principal rule out this possibility.

Therefore, $a_i(y_i) \to -\infty$. Of course, we still can define $\tilde{w}_i$ by the same formula above. Arguing as before, we conclude that we still get a bubble from $\{\tilde{w}_i\}_{i=1}^\infty$, still denoted by $\tilde{v}_\infty$, $D_\infty = D^2 \cup (\mathbf{R}^1 \times S^1) \to \widetilde{M}'$. But the target $\widetilde{M}'$ lying on the left end of $\widetilde{M}$, and $\lim_{s\to\infty} v_\infty(s,t) = x(t)$ in the right end of $\widetilde{M}'$. For simplicity, assume that there is no further splitting of the target. (This follows form our assumption that there is only one bubble if we also count "connecting bubbles".) Then as before, $\tilde{w}_i|_{\Sigma_i - \{y_i\}}$ is locally C^∞- convergent to $\tilde{w}_\infty|_{\mathbf{R}^1\times S^1 - \{y_\infty\}}$ (again assume first that $|y_i| < c$ and use deformation as before to deal with general case), and along the end $D(y_\infty) - \{y_\infty\} \simeq \mathbf{R}^1 \times S^1$,

$$\lim_{s\to\infty} \tilde{w}_\infty(s,t) = x'(t) = x(t) \in \widetilde{M}_- = \widetilde{M}'_+.$$

To see that there is at least one more component of the domain in the limit of $[\tilde{u}_i]$, we note that each u_i connects $x_- \in \widetilde{M}_-$ to $x_+ \in \widetilde{M}_+$, therefore, there exists s_i such that $\tilde{u}_i|_{(-\infty,s_i]\times S^1}$ lies on the "left" of $\tilde{u}_i(y_i)$. Now $\tilde{u}_{i,-R} = \tilde{u}_i|_{(s_i-R,s_i)\times S^1}$ is C^∞-convergent to $\tilde{u}_{\infty,-R}$ after identifying $(s_i - R, s_i) \times S^1$ with $(-R, 0) \times S^1$. We get $\tilde{u}_{\infty,-\infty} = \cup_R \tilde{u}_{\infty,-R} : (-\infty, 0) \times S^1 \to \widetilde{M}'$. We only need to show that for some R, $\tilde{u}_{\infty,-R}$ is not constant map. However, since $\tilde{u}_i$ asymptotically approximates to x_- with exponential decay as s tends to $-\infty$. More precisely, we have $|a_i(s,t) - (cs + d_i)| < e^{-k_i s}$ for some $k_i > 0$. Note that $c = -\int_{S^1} x_-^* \lambda \neq 0$ is the same for all i. Therefore we can replace $\tilde{u}_1(s,t)$ by $\tilde{u}_i(s + s_i, t)$ with some very negative s_i such that $|d\tilde{u}_i|_{[0,R]\times S^1}| > \epsilon > 0$ for some fixed ϵ. Then the above limit $\tilde{u}_{\infty,-\infty}$ is not constant.

This proves the lemma. Note that the component $\tilde{u}_{\infty,-\infty}$ of the limit could come from a closed orbit, i.e. it is a trivial principal components. However, in this case, there is a bubble component lying in the same component of the target.

We remark that in the general case with multi-bubbling, the same proof above proves that each of bubbles lie on some new component of targets which lie on the left of the original $\tilde{M}$. Moreover, there is at least one more principal component lying on the the new "left" component. In particular, in the "new" component of the target, where the first top bubble lies on, there are at least two connected components of the domain of the limit.

QED

Remark 3.1 *Some remark on the special role played by the marked point* (y_i, w_i) *in the above lemma and some related issue is in order. Recall that* y_i *is the point where* $|du(y_i)| \to \infty$ *and* w_i *is the point lying on the cirle of radius* $\frac{1}{|du(y_i)|}$ *measured in the standard metric on* $\mathbf{R} \times S^1$*. In the process of bubbling we bring* (y_i, w_i) *into the standard point* $(0, 1)$ *in* S^2*. On one hand, the point* w_i *will be used to determine the side of bubbling at each stage, on the other, it will also determine two marked lines on the two ends of* Σ_∞ *joint at the double point of* Σ_∞*. Since the two components* v_∞ *and* u_∞ *of the limit approach to a closed orbit* $\{x\}$*, these two marked lines will specify the base point* $1 \in S^1$ *and hence we get a particular paremetrized* $x : S^1 \to M$ *in* $\{x\}$*. However, in the bubbling one can make arbitary choices for* w_i *the the cirle. This implies that in the compactification below, we can use fixed parametrized closed orbits as asymptotic limit to which bubble components, and hence the adjecent principal components, approach along their parametrized ends. On the otherhand, the different* S^1*-paramerizations associated to each of such ends contributs an one dimensional moduli to the domain of the limit stable map. An equivalent way to think about this is to fix an* S^1*-parametrization for each of such ends of the limit curve. Then we can not fix the parametrizations of limit closed orbits anymore.*

Note that in the case of splitting of principal components, as the maked lines are already fixed a priori, clearly, all elements in $\{x\}$ *may appear in the limit.*

Remark 3.2 *The proof of this lemma can be used the deduce some furth restrictions on the possible domains of stable maps, which appeared in the compactifiction of* $\mathcal{M}(x_-, x_+, J)$ *(meaning as a limit of some sequence of elements in* $\mathcal{M}(x_-, x_+, J)$*). There are two general requirements. The first one is that the maximum principal for the a-component of the stable map must hold (as well as the closed related orientation consideration should be incorperated). The second is that there is no loop in the set of components of a domain. Applying these two requirements to the case that there are two connected components of a stable map lying same component of the target with one ordinary double point joint the two components of the domain, one conclude that each connected component has at least one end lying on the positive end of the component of the target. Starting from this, inductively one can prove that in the rightmost (positive) component of the target, there are at least two closed orbits on the positive end of the component of the target, which appeared as the asymptotic limits of the stable map. However, the orintation consideration as in the proof of the last lemma (or the maximum principal plus gluing), implies that this is impossible.*

Therefore, we conclude that all double points of a stable map in the compactification are ends. The same consideration also implies the following simple picture one the structure of the components of a stable map appeared in the compactification. Starting from the leftmost component whose "left" asympototic end is x_-, there exist one and only one end of this component, along which the component approaches to a closed orbit x_1 on the positive end $M_{+,1}$ of component of the target. It is easy to see that all the other ends of the components must lie on the negative end of the component of the target. In this case, since we are already in the leftmost component, this is impossible. However, this can happen in general case and we will use this to do induction in a moment. If the next adjecent component lying on the adjecent component of the target, we are in the same position as before and we can inductively go further. We now show that this must be the case. Otherwise, the new component still stay in the same component of the target, then the induced orientation on x_1 from the two adjecent components are the same, which is a contradiction. We conclude that there is a chain of components (should be called principal components), each lying on different but adjecent components of the target and each connecting two closed orbits on the two different ends of the component of the target. As mentioned above, for each of the principal components, all the other ends (if there are any) must lie on the negative end of the component of the target by maximum principle and gluing.

To get a complete picture, we need to know the behavior of those adjecent components to those negative ends of, say, a typical principal component. The orientation consideration implies that each of such components must lie in the left adjecnt component to the component of the target, on which the principal component lies. The maximum principle and gluing implies that the end at which the principal component and the new adjecent component joint together is the only positive end for the new component. Now we are in the position of induction and we get a very simple structure on the components of a stable map which appears as a limit map. Namely, each component of a stable limit connecting map has only one positive end and possibly many negative ends without any ordinary double points. Starting form the (only) rightmost end x_+, all components of the stable map form a tree pointed to negative a-direction.

It follows from this that for each intermediate closed orbit x, which appears as an end of the limit stable map connecting x_- and x_+, $\int x^*\lambda$ is bounded above by $\int x_+^*\lambda$ and bounded below by the minimum of $\int x_-^*\lambda$ and ϵ, the lower bound of the E_λ-energy of non-trival bubbles. This is used in the proof of Lemma 3.3.

To prove the compactness in general, as in the usual Gromov-Witten theory or Floer homology, there are three steps (i) formation of all bubbles which lie on the top of the bubble tree; (ii) local convergence of the sequence of $\{\tilde{u}_i\}_{i=0}^{\infty}$ along the base, including splitting or degeneration of principal components; (iii) formation of the intermediate bubbles and related "zero bubbling" along connecting necks. Most of analytic part of the proof for these are the analogy to the symplectic case, except the two statments concerning the exponential decay of a bubble along its non-removable singularity and and the behavior of "connecting neck" along the non-compact **R**-direction, detailed in Sec.4. We will only outline the those parts whose proof are similar to the symplectic case.

To do the step (i), we proceed inductively as in the usual symplectic case. The proof of the above lemma serves as the staring point of the induction. During the formtion of the first bubble, the domain of $\tilde{u}_i$ is deformed into $(\Sigma_i; y_i^1, w_i^1)$, where y_i^1, w_i^1 are the maked points denoted by y_i, w_i in the previous lemma. But we think Σ_i as $\mathbf{R}^1 \times S^1$ with a small disc centered at y_i removed, then gluing back a portion of a cylinder, $[0, R_i] \times S^1$ with a half sphere attached. In this model of $(\Sigma_i; y_i^1, w_i^1)$, the maked points y_i^1, w_i^1 becomes the standard points $0, 1$ in the half sphere. Here $R_i = \frac{1}{\alpha}$ and α_i is the deformation parameter in the Lemma before. The target $\tilde{M}$ originally has three marked sections $-\infty, +\infty, 0$. We introduce a new marked section $z_i^1 = a_i(y_i)$, where a_i is the second factor of $\tilde{u}_i$. As proved above, $z_i^1 < 0$ and $|z_i^1 - 0| \to \infty$ as $i \to \infty$.

We then check that if $|d\tilde{u}_i|$ measured in the induced metric in the new deformed domain is uniformly bounded. Assume that is it not. Since the injective radius of these new domains are bounded below, we can repeat the process before to produce second bubble by introducing new marked points y_i^2, w_i^2 in the domain and marked section z_i^2 in the taget which play the same role as y_i^1, w_i^1 and z_i^2 in the formation of the first bubble. As each bubble has a minimal amount of E_λ-energy bounded below, this process will stop after finite steps. We end up with a deformed new domain $(\Sigma_i; y_i^1, w_i^1, y_i^2, w_i^2, \cdots y_i^k, w_i^k)$ of $\tilde{u}_i$. As above, we think it as $\mathbf{R}^1 \times S^1$ with k-small disc centered at $y_i^j, j = 1, \cdots k$ removed, then gluing back a portion of a cylinder, $[0, R_i^j] \times S^1$ with a half sphere attached. As before the maked points y_i^j, w_i^j in $(\Sigma_i; y_i^j, w_i^j; j = 1, \cdots, k)$, becomes the standard points $0, 1$ in these k-half spheres. The target $\tilde{M}$ of $\tilde{u}_i$ now has marked points $-\infty, +\infty, 0, z_i^j, j = 1, \cdots, k$.

Now $|d\tilde{u}_i|$ measured in the induced metric in the new deformed domain is uniformly bounded. Let $D_{i,R}^j, j = 1, \cdots, k$, be one of the k half spheres centered at $0 = y_i^j$ with a portion of a cylinder of length R attached in the deformed domain Σ_i and $D_{i,R}$ be their union. We will use $B_{i,R}$ to denote the subset of Σ_i obtained by removing a small disc around each of those y_i^j which produces a "top" bubble, and then gluing back a cylinder of length R. Then for any fixed R, $\tilde{u}_i|_{D_{i,R}}$ is C^∞-convergent. By letting $R \to \infty$, we obtained all top bubbles.

On the other hand, by restricting $\tilde{u}_i$ to part of $B_{i,R}$ of, say, length R and shiftng the target with a suitable constant, we get the local convergence along the "base" after letting R tend to infinity. Note that in the local convergence of the base, the domain may splitting further into broken connecting maps. It is possible that only one of the two ends of some component of such a broken connecting map approaches to a cloed orbit, the other is just a double point. Note also that during the process of these local convergenes and bubbling, the target also gets split into severl components. For example in the case that each of the distances between these k marked sections $z_i^j, j = 1, \cdots, k$ tends to infinity, the target of the limit has at least $k + 1$ components. This essentially finishes the first two steps (i) and (ii).

It may happen that for some of $B_{i,R}$, the limt of the local convergence is only a constant map. In oder to obtain a meaningful limit along the "base", one has to show it is possible to get a sequence of consective non-trivial limit connecting x_- and x_+. The key point to prove this is to observe that one can have isoperemetric inequality and monetonicity lemma for each $\tilde{u}_i$ projecting to ξ in a small neighbourhood of each point of $\tilde{M}$ as in the usual symplectic case. Now since each $\tilde{u}_i$ connects x_- and x_+, and approaches to some of closed orbits along the ends of the "base", its image projecting to ξ is not very small. This implies that the non-trivial limit of above local convergence can be obtained. The of the analogy argument in symplectic case is used to produce intermediate bubbles, which can be found in [L?]. We refer the readers to the detail there there, which can be easily adapted here.

To do the step(iii), we define the potential "connecting bubble" $C_{i,R} = \Sigma_i \setminus D_{i,R} \cup B_{i,R}$ for fixed R. Each componet $C^k_{i,R}$ of $C_{i,R}$ is a sphere with several small discs removed and cylinder attached, and connects the components of $D_{i,R}$ and $B_{i,R}$. We may asssume that $\lim_{R\mapsto\infty}\lim_{i\mapsto\infty} E_\lambda(\tilde{u}_i|_{C_i}) \neq 0$. Then we get those intermediate connecting bubbles by local convergence of $\tilde{u}_i|_{C^k_{i,R}}$ with $R \to \infty$. As mentioned above isoperemetric inequality and monotonicity lemma for $\tilde{u}_i$ projecting to ξ can be used to produce non-trivial connecting bubbles.

After this is done, we have $\lim_{R\mapsto\infty}\lim_{i\mapsto\infty} E_\lambda(\tilde{u}_i|_{T_i}) = 0$, where $T_i = \Sigma \setminus (B_{i,R} \cup D_{i,R} \cup C_i)$, i.e. there is no E_λ-energy loss any more. We have got the full limit of the sequence $\tilde{u}_i$ along the compact direction. This is the projection of the sequence to the contact manifold M is already weakly convergent to the projection of the limit map so far obtained.

To get the full limit along the non-compact $bf R$-direction, we observe that since there is no E_λ-energy loss anymore, given any two of ends of any of above three parts, if presumely they should joint together in the domain acording the above convergence scheme, but they apporach to two closed orbits which lie on different ends of the target (maybe in the different component of the target also), then the two closed orbits are the same, and we get trivial connecting map between them (maybe passing through several components of the target) as part of the limit. Note that only in the case there is already some non-trivail component lying on some component of the target, we may have to introduce this kind of trivial connecting maps in the component in order to get a connected stable map. Therefore, the limit map so obtained is really a stable map defined before.

Finally, we note that in the next section we will prove that when R and i large enough, each component T^k_i of T_i, whose domain is equivalent to $[-R^k_i, +R^k_i] \times S^1$, is exponetially close to the trivial $\tilde{J}$-holomrphic map coming from some closed orbit x when T^k_i approaches to x.

QED

- **Virtual co-dimension of the boundary of $\overline{\mathcal{M}}(x_-, x_+; J)$:**

Theorem 3.2 *The virtual co-dimension of the boundary components of $\overline{\mathcal{M}}(x_-, x_+; J)$ is at least one. In fact, the co-dimension of the stratum of broken connecting*

$\tilde{J}$-holomorphic maps of two elements is one , and co-dimension of any stratum whose elements contain bubble component is at least two.

proof

The proof of this theorem depends on the index formula, which will be proved in [L3].

Let $[\tilde{u}]$ be a typical element in the stratum. It is sufficient to consider the follow two cases:

(i) The domain Σ of u is $\Sigma_1 \cup \Sigma_2$ joint together at one of the ends of Σ_1 and Σ_2. Each $\Sigma_i, i = 1, 2$ is S^2 with two marked points $-\infty$ and $+\infty$ treated as ends, and we identify $\Sigma_i \setminus \{end\}$ with $S^1 \times \mathbf{R}$ to give two marked lines on Σ. The target U of $\tilde{u}$ is $\tilde{M}_1 \cup \tilde{M}_2$ joint at one of their ends. $\tilde{u}_1$ connects a closed orbit $x_{-,1}$ on $\tilde{M}_{-,1}$ and another closed orbit x on $\tilde{M}_{+,1} = \tilde{M}_{-,2}$, and $\tilde{u}_2$ connects the closed orbit x and another closed orbit $x_{+,2}$ on $\tilde{M}_{+,2}$.

Note that $x \neq x_{-,1} \neq x_{+,2}$, and $x \neq x_{+,2}$. There are five dimensional symmetries for each element $[\tilde{u}]$ is the above stratum, two dimension coming from the $\mathbf{R}^1$-translations on each factor of the target and two dimensional $\mathbf{R}^1$-translations on each factor of the domain together with an S^1-action on the domain. We will slice out the S^1-action first. Let $\tilde{\mathcal{M}}(x_-, \{x\}, \{x_+\}; J)$ be the moduli space of marametrized broken connecting $\tilde{J}$-maps of two elements as above. But we fix a parametrized x_- and alow x and x_+ vary in their equivalent classes. The dimension of the symmetry group of the moduli space is 4. It follows from the index formula in [L3] that the dimension of $\tilde{\mathcal{M}}(x_-, \{x\}, \{x_+\}; J)$ is same as the dimension of $\tilde{\mathcal{M}}(x_-, \{x_+\}; J)$ plus one, due to the one dimensional possible choices of the element $x \in \{x\}$. Now a direct dimension counting on the symmetries shows that in this case the codimension the the boundary component of $\overline{\mathcal{M}}(x_-, x_+; J)$ is one.

(ii) The second case corresponds to the case that there is only only one bubbling as described in Lemma 3.4. There are two different subcases: (1) both of the principal components are non-trivial; (2) the "new" principal components is trivial. In the case (1), along the princial component, as parametrized map, there are three different possible parametrized closed orbits as asympotic limit along ends, but is the case (2), there are only two of such closed orbits. On the other hand, the dimension of the symmetry group of the two components lying in the "left" component of the target (not counting the S^1-action) is 6 in the case (1) and 5 in the case (2). Note that in the case (1) there are two connected components in the "new' components of the target, while in the case (2) there is only one according to our convention introduced before. Again index formula in [L3] together with a direct dimension counting argument gives the desire conclusion in this case.

QED

4 Exponential Decay Estimate

We have proved a version of compactness theorem for the moduli space of stable $\tilde{J}$-holomorphic maps in last section. The result is not quite completed for its own ppurpose as well as for later applications. As we have shown before that a sequence of $\tilde{J}$-holomorphic maps may develop bubbles and split into broken connecting $\tilde{J}$-holomorphic maps. Unlike the usual Gromov-Floer theory, these bubbles always have unremovable singularities. We showed before that along the ends of singularities, the bubbles approach to some closed orbits. For the purpose of moduli cycles in [L1], it is important to know the rate of the $\tilde{J}$-holomorphic maps approach to closed orbits either along their ends or along the ends of the singularities of the bubbles. One of the main results of this section is to prove that the rate of the approximation is exponential. When $\dim M = 3$, this is proved by Hofer, Wysocki and Zehnder in [HZW]. When M has an S^1-symmetry, this is proved by Li-Ruan in [LiR]. We remark that the extra assumption of [LiR] considerably simplified the analysis here. On the other hand, the general case, even in $\dim M = 3$, the argument in [HZW] is quite involved. It turns out that the method of [HZW], suitably modified, can be extended to the general case. We will carry out this generalization somewhere else. In this section, we will give a more abstract and a simpler proof.

To motivate the second main result of this section, we note that one of the important ingredients of the proof of compactness of the moduli space in the usual Gromov-Floer theory is an explicit description about the behavior of the "connecting neck" near bubble point.

In our case, it is necessary to know that the behavior of the "connecting necks" near the "connecting" closed orbit when a family of $\tilde{J}$-holomorphic maps develop, say, a bubble approaching to the "connecting" closed orbit, or split into a broken $\tilde{J}$- holomorphic maps of two elements joints at the closed orbit. More precisely, if

$$\tilde{v}_i = \tilde{u}_i|_{[-l_i,l_i]\times S^1} : [-l_i, l_i] \times S^1 \to \tilde{M} = M \times \mathbf{R}$$

is the "neck" part of $\tilde{u}_i$ such that the M-projection v_i is close to the closed orbit $x(t)$ with $c = \int_{S^1} x^*\lambda$. We claim that $\tilde{v}_i$ is essentially the same as the trivial map $(s,t) \to (x(t), cs) \in M \times \mathbf{R}$ restricted to $[-l_i, l_i] \times S^1$. In particular, the length of $\mathbf{R}^1$- projective of $\tilde{v}_i$ differs from $2c \cdot l_i$ by at most a fixed small constant. Note that when $i \to \infty, l_i \to \infty$. It turns out that this statement plays an important role in the compactness theorem. Recall that we have required that in the definition of stable map, there is no unstable trivial connecting maps appeared as components. The justification of this is based on the above statement.

Let $x(t)$ be a closed orbit. $\tilde{u} = (u, a), \tilde{w} = (w, b)$ are two $\tilde{J}$-holomorphic connecting maps: $\mathbf{R}^1 \times S^1 \to \tilde{M}$ such that $\lim_{s\to+\infty} u(s,t) = x(t) = \lim_{s\to-\infty} w(s,t)$. Assume that $\lim_{s\to-\infty} u(s,t) = x_-(t)$ and $\lim_{s\to+\infty} w(s,t) = x_+(t)$. Let $\tilde{v}_i^* = (v_i^*, f_i^*) : \mathbf{R}^1 \times S^1 \to \tilde{M}$ be a sequence of $\tilde{J}$- holomorphic maps connecting $x_-(t)$ and $x_+(t)$ and locally convergent to $\tilde{u} \cup \tilde{w}$. Hence, $\lim_{s\to+\infty} u(s,t) = \lim_{s\to-\infty} w(s,t) = x(t)$ of some closed orbit. Note that the target $\tilde{M}$ of $\tilde{u}$ and $\tilde{w}$ should be thought as two different spaces joint together at their ends. We will only prove our results for this particular case. It is easy to see that the corresponding results for the case that $\tilde{v}^*$ produces only one bubble can be proved in an exactly the same way and the result for the general case can be obtained by a simple combination of these two cases.

The assumption that $\tilde{v}_i^*$ is locally C^∞-convergent to $\tilde{u}\cup\tilde{w}$ implies that there exist $n_{i,j} \in \mathbf{R}$, $m_{i,j} \in \mathbf{R}$, $j = 1, 2$ such that $\tilde{v}_n^*(s + n_{i,1}, t) + (0, m_{i,1})$ is C^∞-convergent to $\tilde{u}(s,t)$ and $v_i^*(s + n_{i,2}, t) = (0, m_{i,2})$ is C^∞-convergent to $\tilde{v}(s,t)$ for any compact subset of $\mathbf{R}^1 \times S^1$.

Now both $\{\tilde{u}(s+n,t)\}_{n=0}^\infty$ and $\{\tilde{v}(s-n,t)\}_{n=0}^\infty$ are locally C^∞-convergent to the trivial $\tilde{J}$-holomorphic map $(s,t) \to (x(t), cs)$, after translations in $\tilde{M}$. We conclude that $\exists N$ such that for any given $\epsilon > 0$, when $s > N$, $|D^\alpha\{u(s,t) - x(t)\}| < \epsilon = \epsilon_\alpha$ and $S < -N$, $|D^\alpha\{w(s,t) - x(t)\}| < \epsilon = \epsilon_\alpha$ for any $|\alpha| \geq 0$, and that $|D^\alpha\{a(s,t) - c\dot{s}\}| < \epsilon = \epsilon_\alpha$, $|D^\alpha\{b(s,t) - c\dot{s}\}| < \epsilon = \epsilon_\alpha$ for any $|\alpha| \geq 1$.

We now define $\tilde{v}_i(s,t) = \tilde{v}_i^*(s + \frac{n_{i,1}+n_{i,2}}{2}, t)$. by the assumption on local convergence of $\tilde{v}_i^*$, $n_{i,1} \to -\infty$ and $n_{i,2} \to +\infty$. Let $l_i = \frac{1}{2}\{(n_{i,2} - n_{i,1} - 2N\}$. Then $l_i \to +\infty$. Then $v_i(-l_i, t) = v_i^*(N + n_{i,1}, t) \to u(N, t)$ and $v_i(l_t, t) = v_i^*(-N + n_{i,2}, t) \to w(-N, t)$.

Lemma 4.1 *When i is large enough, for any $s \in (-l_i, l_i)$, $|D^\alpha\{v_i(s,t) - x(t)\}| < 2\epsilon$, $|\alpha| \geq 0$ and $|D^\alpha\{f_i(s,t) - cs\}| < 2\epsilon$, $|\alpha| \geq 1$.*

proof

Since the proof of the two statements are similar, we will only prove the first one. Assume that the first statement is not true. then there exists a sequence $(s_i, t_i) \in (-l_i, l_i) \times S^1$, $i \to \infty$, such that $|D^\alpha\{v_i(s_i, t_i) - x(t_i)\}| > 2\epsilon$. If $|s_i - (-l_i)|$ or $|s_i - l_i|$ are bounded, say $|s_i - (-l_i)|$ is bounded, then $v_i(s_i + s, t), s \in (-\delta, \delta)$ is C^∞-convergent to $u(\underline{N} + s, t)$ for some $\underline{N} > N$ and $s \in (-\delta, \delta)$, which implies that

$$|D^\alpha\{v_i(s_i, t) - x(t)\}| < 2\epsilon$$

when i is large enough. This is a contradiction. Hence we may assume that both $|s_i - (-l_i)|$ and $|s_i - l_i| \to \infty$.

Then $\tilde{v}_i(s_i + s, t)$ is still C^∞-convergent for any $(s,t) \in [-R, R] \times S^1$, with fixed R. Let $R \to \infty$ and patch all the local limit together, we get a $\tilde{J}$- holomorphic map $\tilde{v}_\infty : \mathbf{R}^1 \times S^1 \to \tilde{M}$ with $E_\lambda(\tilde{v}_\infty) = 0$. This implies that $v_\infty(s,t) = x(t)$. Therefore, $|D^\alpha(v_i(s_i, t) - x(t))| < \epsilon$ when i large enough. This is a contradiction again.

QED

To state one of our main results, we define

$$\tilde{v}_{i,+}(s,t) = (v_i(s - l_i, t), f_i(s - l_i, t) - f(-l_i, 0) + a(N, 0))$$

and

$$\tilde{v}_{i,-}(s,t) = (v_i(-s + l_i, t), f_i(-s + l_i, t) - f(l_i, 0) + b(-N, 0)).$$

Then $\tilde{v}_{i,+}(0,0) \to (u(N,0), a(N,0))$, and $\tilde{v}_{i,-}(0,0) \to (w(-N,0), b(-N,0))$.

• **Local Coordinate near x(t)**:

The λ-period of $x(t)$ is $\int_{S^1} x^*\lambda dt = c$. We have $\frac{dx}{dt} = c\dot{X}_\lambda(\alpha(t))$. By rescaling the parameter (s,t), we may assume that $c = 1$. Let τ be the minimal period of $x(t)$, i.e. $\tau > 0$ is the minimal number such that $x(t+\tau) = x(t)$. Under this assumption, given any point $z = x(t), t \in [0,\tau)$, we assign its θ-coordinate $\theta = \theta(z) = t$. For simplicity, we will assume further that $\tau = 1$. Hence $\theta \in S^1 = \mathbf{R}/\mathbf{Z}$, and $x(\theta) = x(t), \theta \in S^1$ is the simple closed orbit. Choose a global basis $\{e_1, \cdots, e_{2n}\}$ for the symplectic bundle $(\xi, d\lambda)|_{x(\theta)}$ such that the map

$$y = \Sigma y_i e_i(x(\theta)) \in \xi \to (\theta, y_1, \cdots, y_{2n}) \in (S^1 \times \mathbf{R}^{2n}, \omega_0)$$

gives rise a isomorphism between the two trivial symplectic bundles $(\xi, d\lambda)$ and $(S^1 \times \mathbf{R}^{2n}, \omega_0)$ over S^1. The local coordinate of M near $x(\theta)$ is define by $(y,\theta) \to exp_{x(\theta)}\Sigma y_i e_i$, where $y = (y_1, \cdots, y_{2n}) \in \mathbf{R}^{2n}$, $\theta \in S^1$. The exponential map is taken with respect to the Riemanian metric $g_{\tilde{J}}$. Note that we may assume that $J|_{\xi|_{S^1}}$ corresponds to J_0 under above iomorphism of the two symplectic bundles over $S^1 = \{x(\theta)\}$.

Let U be a small tube neighborhood of x in M. With the above coordinate (y,θ), then at any point $z \in U$,

$$T_zM = \mathbf{R}\{\frac{\partial}{\partial\theta}\} \oplus \mathbf{R}\{\frac{\partial}{\partial y_1}, \cdots, \frac{\partial}{\partial y_{2n}}\} = \mathbf{R}X_\lambda \oplus \xi_z.$$

Since at $y = 0$, $\xi|_{y=0} = \mathbf{R}\{\frac{\partial}{\partial y_1}, \cdots, \frac{\partial}{\partial y_n}\}_{y=0}$, the projection $d\pi_y : T_zM \to \mathbf{R}\{\frac{\partial}{\partial y_1}, \cdots, \frac{\partial}{\partial y_{2n}}\}_z$ when restricted to ξ_z, is an isomorphism, when $|y|$ is small enough. Here $z = (y,\theta)$. We may assume that any $z \in U$ has this property. Then we can find $e_i = e_i(z)$ such that $d\pi_y(e_i) = \frac{\partial}{\partial y_i}$. Since $d\pi_y(\frac{\partial}{\partial\theta}) = 0$, $\frac{\partial}{\partial\theta} \notin \xi_z$. Hence $\mathbf{R}\{\frac{\partial}{\partial\theta}\} \oplus \xi_z = T_zM$.

For the application later, we need to compare e_i with $\frac{\partial}{\partial y_i}$ and X_λ with $\frac{\partial}{\partial\theta}$. Let $e_i = \Sigma_{i=1}^{2n}\alpha_{i,j}(z)\frac{\partial}{\partial y_i} + \alpha_{i,0}(z)\frac{\partial}{\partial\theta}$, $X_\lambda = \Sigma_{i=1}^{2n}X_i(z)\frac{\partial}{\partial y_i} + X_0(z)\frac{\partial}{\partial\theta}$. Here $\alpha_{i,j}$ and X_i are functions defined on $\mathbf{R}^{2n} \times S^1 = \{(y,\theta)\}$. Fix i, since $e_i(0,\theta) = \frac{\partial}{\partial y_i}$,

$$\begin{aligned} e_i(y,\theta) - \frac{\partial}{\partial y_i} &= e_i(y,\theta) - e_i(0,\theta) \\ &= (\frac{\partial}{\partial\theta}, \frac{\partial}{\partial y_1}, \cdots, \frac{\partial}{\partial y_{2n}})\{\int_0^1 \frac{d}{d\tau}\begin{bmatrix} \alpha_{i,0}(\theta,\tau y) \\ \alpha_{i,1}(\theta,\tau y) \\ \vdots \\ \alpha_{i,2n}(\theta,\tau y) \end{bmatrix} d\tau\} \\ &= (\frac{\partial}{\partial\theta}, \frac{\partial}{\partial y_1}, \cdots, \frac{\partial}{\partial y_{2n}})(\int d\alpha_i(\theta,\tau y)d\tau)\begin{bmatrix} y_1 \\ \vdots \\ v_{2n} \end{bmatrix}. \end{aligned}$$

Here $d\alpha_i(\theta, y) = [\frac{\partial \alpha_{i,j}}{\partial y_k}(\theta, y)]$ is the $(2n+1)\times(2n+1)$ matrix where the (j,k)th element is $\frac{\partial \alpha_{i,j}}{\partial y_k}(\theta, y)$. Similarly,

$$X\lambda(y,\theta) - \frac{\partial}{\partial\theta} = (\frac{\partial}{\partial\theta}, \frac{\partial}{\partial y_1}, \cdots, \frac{\partial}{\partial y_{2n}}) \int_0^1 dx(\theta, \tau y) d\tau \begin{pmatrix} y_1 \\ \vdots \\ y_{2n} \end{pmatrix},$$

where $dx(\theta, y)$ is the $(2n+1)\times(2n+1)$ matrix whose (j,k) element is $\frac{\partial X_j}{\partial y_k}(\theta, y)$. Not that both matrices $d\alpha_i$ and dx has uniformly bounded norm for $(y,\theta) \in U$. This proves

Lemma 4.2 *For any $(y,\theta) \in U$, $\exists$ constant C such that*

$$|e_i(y,\theta) - \frac{\partial}{\partial y_i}| < C\dot{|}y|, \quad |X_\lambda(y,\theta) - \frac{\partial}{\partial\theta}| < C\dot{|}y|.$$

In the (y,θ,a)-coordinate for $U \times \mathbf{R} \subset \tilde{M}$, we write $\tilde{u}(s,t) = (u(s,t), a(s,t))$ and $u(s,t) = (y_u(s,t), \theta_u(s,t))$. If there is no confusion, we will simply ommit the subscript u in y_u and θ_u. Similarly, we write $w(s,t) = (y_w(s,t), \theta_w(s,t))$ and $v_i(s,t) = (y_{v_i}(s,t)), \theta_{v_i}(s,t)$ in the (y,θ)-coordinate.

Lemma 4.3 *Let $\pi = \pi_\xi : TM = \mathbf{R}\{X_\lambda\} \oplus \xi \to \xi$ be the projection. Given any $v \in TM$, if*

$$\pi(v) = \Sigma_{i=1}^{2n} c_i \frac{\partial}{\partial y_i} + c_0 \frac{\partial}{\partial\theta} = \Sigma_{i=1}^{2n} d_i e_i,$$

then $c_i = d_i, i = 1, \cdots, 2n$.

Proof

$\pi(v) = v - \lambda(v) X_\lambda$. Let

$$X_\lambda = \Sigma_{i=0}^{2n} X_i \frac{\partial}{\partial y_i} + X_0 \frac{\partial}{\partial\theta}$$

and

$$v = \Sigma_{i=1}^{2n} v_i \frac{\partial}{\partial y_i} + v_0 \frac{\partial}{\partial\theta}.$$

Then

$$\begin{aligned} \pi(v) &= \Sigma_{i=1}^{2n}(v_i - \lambda(v)\cdot X_i)\frac{\partial}{\partial y_i} + (v_0 - \lambda(v)X_0)\frac{\partial}{\partial\theta} \\ &= \Sigma_{i=1}^{2n}(v_i - \lambda(v)\cdot X_i)e_i + (v_0 - \lambda(v)X_0)\frac{\partial}{\partial\theta} \\ &+ \Sigma_{i=1}^{2n}(v_i - \lambda(v)X_i\cdot(\frac{\partial}{\partial y_i} - e_i). \end{aligned}$$

Now since $\pi_y(\frac{\partial}{\partial y_i} - e_i) = \frac{\partial}{\partial y_i} - \frac{\partial}{\partial y_i} = 0$,

$$\frac{\partial}{\partial y_i} - e_i \in \ker \pi_y = \mathbf{R}\{\frac{\partial}{\partial\theta}\}.$$

Therefore

$$\Sigma_{i=1}^{2n}(v_i - \lambda(v)X_i)(\frac{\partial}{\partial y_i} - e_i) \in \mathbf{R}\{\frac{\partial}{\partial\theta}\}$$

and

$$\pi(v) = \Sigma_{i=1}^{2n}(v_i - \lambda(v)X_i)e_i,\ \mathrm{mod}(\mathbf{R}\{\frac{\partial}{\partial\theta}\}).$$

But $\pi(v), e_i \in \xi$ and $\frac{\partial}{\partial\theta} \notin \xi$. This implies that $\pi(v) = \Sigma_{i=1}^{2n}(v_i - \lambda(v)X_i)e_i$.

QED

• **Equation in the local coordinate**:

We only write the equation for $\tilde{u}$. Same expression is also applicable to $\tilde{w}$ and $\tilde{v}_i$. That $\tilde{u}$ is $\tilde{J}$- holomorphic is equivalent to:

$$\begin{cases} a_s & = \lambda(u_t) & (a) \\ a_t & = -\lambda(u_s) & (b) \\ \pi(u)\circ du\circ i & = J(u)\pi(u)\circ du & (c) \end{cases}$$

Let $M(y,\theta)$ be the $2n\times 2n$ matrix for the $d\lambda$- compatible almost complex structure $J(y,\theta)$ with respect ot the basis $\{e_1,\cdots,e_{2n}\}$. We will assume that $M(y,\theta) = J_0$, the standard constant complex structure on $\mathbf{R}^{2n}$. That is $J_0(e_i) = e_{i+n}$ and $J_0(e_{i+n}) = -e_i$, $1\le i\le n$. As pointed out in [HZW], the proof of the statements below for general M can be reduced to this case. For our purpose of this paper, we can even assume that this is really true as we can make choice of J. The eqation (c) is equivalent to $\pi(u_s) + J(u)\pi(u_t) = 0$. In local coordinate we have

$$\begin{aligned} \pi(u_s) &= \Sigma_{i=1}^{2n}\{(y_i)_s - \lambda(u_s)X_i\}e_i \\ \pi(u_t) &= \Sigma_{i=1}^{2n}\{(y_i)_y - \lambda(u_t)X_i\}e_i. \end{aligned}$$

Hence,

$$(y_s - \lambda(u_s)Y) + M(y_t - \lambda(u_t)Y) = 0.$$

Equivalently,

$$y_s + My_t + (a_t - a_s\cdot M)\cdot Y = 0.$$

Here $y = \begin{bmatrix} y_1 \\ \vdots \\ y_{2n}\end{bmatrix}$ and $Y = \begin{bmatrix} X_1 \\ \vdots \\ X_{2n}\end{bmatrix}$, and $M = J_0$.

We have shown that

$$Y(y,\theta) = \{\int_0^1 dY(\tau y,\theta)d\tau\}\begin{pmatrix} y_1 \\ \vdots \\ y_{2n}\end{pmatrix}$$

and $dY(y,\theta)$ is the $2n\times 2n$ matrix whose (j,k)-element is $\frac{\partial X_j}{\partial y_k}$.

Denote $\int_0^1 dY(\tau y, \theta) d\tau$ by $DY(y, \theta)$. Then

$$y_s + My_t + \{(a_t - a_s M) \cdot DY\} \cdot y = 0.$$

Denote $\{a_t - a_s M\} \cdot DY(y(s,t), \theta(s,t))$ by $S(s,t)$. We define $S_\infty = -J_0 \cdot dY(0,t)$.

Lemma 4.4 *When $s > N$, $|S(s,t) - S_\infty(s,t)| < C \cdot \epsilon$ and $|S_s(s,t)| < C \cdot \epsilon$ for the given ϵ and some constant C. Same conclusion for w and v_i when $s < -N$ or $s \in (-l_i, l_i)$ respectively.*

Proof:

We only prove the statement for u.

When $s > N$,

$$\begin{aligned} |D_s\{u(s,t) - x(t)\}| &= |D_s\{(y(s,t), \theta(s,t)) - (0,t)\}| \\ &= |D_s(y(s,t), \theta(s,t))| < \epsilon. \end{aligned}$$

Note that in the (y, θ)-coordinate, $x(t) = (0,t)$ since $c = 1$. Similarly, when $s > N$,

$$|D_s\{a_t(s,t) - \frac{\partial}{\partial t}(cs)\}| = |D_s a_t(s,t)| < \epsilon$$

and

$$|D_s\{a_s(s,t) - \frac{\partial}{\partial s}(cs)\}| = |D_s a_s(s,t)| < \epsilon.$$

This implies that $|D_s S(s,t)| < C \cdot \epsilon$ for some constant C depending only on $||DY(y,0)||_{C^1}$ on U.

When $s > N$, $|a_t(s,t)| = |D_t(a(s,t) - cs)| < \epsilon$ with $c = 1$ and $|a_s(s,t) - 1| = |D_s(a(s,t) - cs)| < \epsilon$, we have

$$|(y(s,t), \theta(s,t)) - (0,t)| < \epsilon.$$

This implies that

$$|S(s,t) - \{-J_0 DY(0,t)\}| < \epsilon.$$

But

$$\begin{aligned} -J_0 DY(0,t) &= -J_0 \int_o^1 dY(0,t) d\tau \\ &= -J_0 dY(0,t) = S_\infty(t). \end{aligned}$$

QED

Lemma 4.5 *$S_\infty(t)$ is a $2n \times 2n$ symmetric metric and all the eigenvalues of the self-adjoint elliptic operator $A_\infty : L_1^2(S^1, \mathbf{R}^{2n}) \to L^2(S^1, \mathbf{R}^{2n})$ defined by $A_\infty : z \to -J_0 \frac{dz}{dt} - S_\infty \cdot z$, are non-zero.*

Proof:

Let Ψ_t be the flow of X_λ. Hence

$$\left\{\begin{array}{l} \frac{d\Psi_t(z)}{dt} = X_\lambda(\Psi_t(x)) \\ \Psi_0(z) = z, \forall z \in M. \end{array}\right. \qquad (*)$$

If $z_0 = (0,0)$ in (y,θ)-coordinate then $\Psi_t(z_0) = \Psi_t(0,0) = (0,t) = x(t)$. Hence $z_0 = \Psi_1(z_0)$ is a fixed point of Ψ_1. Note that the flow Ψ_t preserves the decomposition $TM = \mathbf{R}\{X_\lambda\} \oplus \xi$, and that along $x(t) = (0,t)$, $X_\lambda = \frac{\partial}{\partial\theta}$ and $\xi = \mathbf{R}\{\frac{\partial}{\partial y_1}, \cdots, \frac{\partial}{\partial y_{2n}}\}$. Differentiating eqaution (*) above, we get

$$\frac{dD\Psi_t}{dt} = DX_\lambda(\Psi_t) \circ D\Psi_t. \qquad (**)$$

Now given $v, w \in \xi_{(0,0)} \subset T_{(0,0)}M$, since Ψ_t preserves $d\lambda = \omega$, We have

$$\begin{aligned} \omega(J(\Psi_t)_*(v), J(\Psi_t)_*(w_0)) &= \\ \omega((\Psi_t)_*(v), (\Psi_t)_*(w)) &= \omega(v,w). \end{aligned}$$

Differentiating this, we get

$$\omega(J(\frac{d}{dt}D\Psi_t)(v), JD\Psi_t(w)) + \omega(JD\Psi_t(v), J(\frac{d}{dt}D\Psi_t)(w)) = 0.$$

Here we used that J is constant along $\xi|_{x(t)}$. Use equation (**), we get

$$\omega(JDX_\lambda(\Psi_t) \circ D\Psi_t(v), JD\Psi_t(w)) + \omega(JD\Psi_t(v), JDX_\lambda(\Psi_t) \circ D\Psi_t(v)) = 0.$$

Let $v_t = D\Psi_t(v), w_t = D\Psi_t(w)$. Then

$$g_J(JDX_\lambda(\Psi_t)(v_t), w_t) = g_J(v_t, JDX_\lambda(\Psi_t)(w_t)).$$

Let $t = 1$, then $\Psi_t(z) = z$ for any $z = (0,\theta)$. It is easy to see that

$$DX_\lambda(0,t) = \left(\begin{array}{ll} dY(0,t) & 0 \\ 0 & 1 \end{array}\right),$$

$$JDX_\lambda(0,0)(v_1) = JdY(0,0)(v_1),$$

and

$$JDX_\lambda(0,0)(w_1) = JdY(0,0)(w_1).$$

This implies that $S_\infty = -J_0 dY(0,0)$ is symmetric. Then general case can be proved by a coordinate change on t. Therefore, $A_\infty = -J_0\frac{d}{dt} - S_\infty : L_1^2(S^1, \mathbf{R}^{2n}) \to L^2(S^1, \mathbf{R}^n)$ is a self-adjoint elliptic operator. We want to show that 0 is not an eigenvalue of A_∞. Given $0 \neq z \in L_1^2(S^1, \mathbf{R}^{2n})$, $A_\infty(z) = 0$ is equivalent to

$$\frac{dz}{dt} = J_0 S_\infty(t) z = dY(0,1)z, \qquad (***)$$

with $z(t+1) = z(t)$. As before let $z_0 = (0,0)$. Then $\Psi_t(z_0) = (0,t)$ in (y,θ)-coordinate. We write

$$D\Psi_t(z_0) = \begin{pmatrix} R(t) & 1 \\ 0 & 1 \end{pmatrix}$$

with respect to the basis

$$\{\frac{\partial}{\partial y_1}, \cdots, \frac{\partial}{\partial y_{2n}}, \frac{\partial}{\partial \theta}\}.$$

The equation (**) implies

$$\frac{dR(t)}{dt} = dY(0,t) \cdot R(t). \qquad (****).$$

If $w(t) \neq 0$ is a solution of (***), then $w(t+1) = w(t)$.

Define $\tilde{w}(t) = R(t) \cdot w(0)$, then (****) implies

$$\frac{d\tilde{w}}{dt} = dY(0,t)\tilde{w}(t).$$

Since $\tilde{w}(0) = R(0)w(0) = w(0)$, we have $\tilde{w}(1) = w(1) = w(0)$, i.e. $w(0)$ is an eigenvalue of $R(1)$ with eigenvalue 1. This implies that $d\Psi_1(z_0)$ has an eigenvector of eigenvalue 1 along ξ_{z_0}. Conversely, if v is an eigenvector of $R(1)$ with eigenvalue 1, then $w(t) = R(t) \cdot v$ solves (***) with $w(1) = R(1) \cdot v = v = R(0) \cdot v = w(0)$. Therefore, we get an eigenvector of A_∞ of eigenvalue 0.

It follows from this and previous lemma that

Lemma 4.6 *There exists a constant $\delta > 0$, such that when N and i large enough, for $\tilde{u}$ and $\tilde{w}$, with $s > N$ or $s < -N$ respectively,*

$$\|(-J_0\frac{d}{dt} - S(s,t)) \cdot z\| \geq 2\delta\|z\|, \forall z \in L^2_1(S^1, \mathbf{R}^{2n}).$$

For $\tilde{v}_i$ with $s \in (-l_i, l_i)$, same conclusion holds.

We denote $-J_0\frac{d}{dt} - S(s,t)$ by $A(s) : L^2_1(S^1; \mathbf{R}^{2n}) \to L^2(S^1; \mathbf{R}^{2n})$. Note that $|A(s) - A^*(s)| = |S - S^*| \leq |S - S_\infty| + |S^* - S^*_\infty| < c \cdot \epsilon$, when $s > N$ or $s < -N$ for $\tilde{u}$ or $\tilde{w}$, or $S \in (-l_i, l_i)$ for $\tilde{v}_i$.

We now establish the exponential decay estimate for the y-components of $\tilde{u}$, $\tilde{w}$ and $\tilde{v}_i$. We will use $y = y(s) = y(s, -) \in L^2_1(S^1; \mathbf{R}^{2n})$ to denote the y-components of $\tilde{u}$, $\tilde{w}$ or $\tilde{v}_i$. Let $g(s) = \frac{1}{2} < y(s), y(s) >$.

Lemma 4.7 *When N and i large enough, for $s > N$ or $s < -N$ for $\tilde{u}$ or $\tilde{w}$, and for $s \in (-l_i, l_i)$ for $\tilde{v}_i$, we have*

$$g^{''}(s) \geq \delta^2 g(s).$$

Proof:

$$\begin{aligned}
g'(s) &= < y'(s), y(s) > . \\
g''(s) &= < y_s, y_s > + < (y_s)', y(s) > \\
&= < A \cdot y, A \cdot y > + < \frac{\partial}{\partial s}(-J_0 \frac{dy}{dt} - S \cdot y), y(s) > \\
&= < A \cdot y, A \cdot y > + < y_s, A^* y > - < S_s y, y > \\
&= 2\|A \cdot y\|^2 + < A \cdot y, (A^* - A) \cdot y > - < S_s y, y > \\
&\geq 2\|Ay\|^2 - C\epsilon\|Ay\| \cdot \|y\| - C\epsilon\|y\|^2 \\
&= \|Ay\|(2\|Ay\| - C\epsilon\|y\|) - C\epsilon\|y\|^2 \\
&\geq \delta\|y\|^2(2\delta - C\epsilon - \frac{C\epsilon}{\delta}) \\
&\geq \delta^2\|y\|^2 = \delta^2 g(s).
\end{aligned}$$

Here we use the fact that C and δ are uniformly bounded for all s and ϵ can be made as small as possible by the suitable choice of s in the lemma.

QED

For $\tilde{u}$ and $\tilde{w}$, since $s \in [N, +\infty)$ or $s \in (-\infty, -N]$, and $g(s) \to 0$ as $s \to \pm\infty$, the above lemma together with the usual elliptic estimate applied to each $[s_i, s_i + 1] \times S^1$ implies that

Lemma 4.8 *The y-component $y(s,t)$ of $\tilde{u}$ satisfies:*

$$\|y(s)\|^2_{L^2} \leq \|y(N)\|^2_{L^2} \cdot e^{-\delta(s-N)}, \quad s > N.$$

Moreover, there exists a constant $C = C_\alpha$, with $|\alpha| \geq 0$, such that

$$|D^\alpha y(s,t)| < C_\alpha \cdot e^{-\delta(s-N)}.$$

Similar conclusion holds for $\tilde{w}$.

To get corresponding estimate for $\tilde{v}_i$, we note that since $\lim_{s \to +\infty} y_u(s,t) = 0 = \lim y_w(s,t)$, we may assume that $\|y_u(N)\|_{L^2} = \|y_w(-N)\|_{L^2}$. This implies that $\|y_{v_i}(-l_i)\|_{L^2}$ is very close to $\|y_{v_i}(l_i)\|_{L^2}$, when i large enough. For simplicity, we may assume that $c_+ = g(l_i) = \|y_{v_i}(l_i)\|^2 = \|y_{v_i}(-l_i)\|^2 = g(-l_i) = c_-$. Let $c = c_+ = c_-$, and denote l_i by l. Define $h(s) = a \cdot (e^{-\delta s} + e^{\delta s})$, with $a = \frac{c}{e^{-\delta l} + e^{\delta l}}$. Then $h(-l) = h(l) = c$, and $h''(s) = \delta^2 h(s)$.

Define $f = g - h$. Then $f''(s) \geq \delta^2 \cdot f(s)$, for $s \in (-l, l)$ and $f(-l) = f(l) = 0$. The maximal principle implies that $f(s) \leq 0, s \in (-l, l)$. Hence $g(s) \leq \frac{c \cdot (e^{-\delta s} + e^{\delta s})}{e^{-\delta l} + e^{\delta l}}$.

Now define $g_+(s) = g(s - l)$, and $g_-(s) = g(l - s)$, $s \in (0, l)$. Then

$$\begin{aligned}
g_+(s) &\leq c \cdot \frac{e^{-\delta(s-l)} + e^{\delta(s-l)}}{e^{-\delta l} + e^{\delta l}} \\
&\leq 2 \cdot c \frac{e^{-\delta s} \cdot e^{\delta l}}{e^{\delta l}} = 2g_+(0) \cdot e^{-\delta s} \\
&= 2c_+ e^{-\delta s}, \qquad\qquad s \in [0, l].
\end{aligned}$$

Similarly, $g_s(s) \leq 2c_- e^{-\delta s}$.

Note that since c_+, c_- are close to $\|y_u(N)\|^2_{L^2}$ and $\|y_w(-N)\|^2_{L^2}$ which are fixed, we get exponential decay of $g_+(s)$ and $g_-(s)$. For the general case when $c_+ \neq c_-$, we have $g_+(s) \leq 2(c_+ + c_- + \epsilon) \cdot e^{-\delta s}$ for some fixed small ϵ when i large enough.

Define $\tilde{v}_{i,+} = \tilde{v}_i(s - l_i, t)$ and $\tilde{v}_{i,-}(s,t) = \tilde{v}_i(l_i - s, t)$, and let y_+, y_- be the corresponding y-components. We have

Lemma 4.9 *When i large enough,*

$$|y_\pm(s,t)|^2_{L^2} \leq 2(\|y_u(N)\|^2_{L^2} + \|y_w(-N)\|^2_{L^2} + \epsilon) \cdot e^{-\delta s}, \quad s \in (0, l_i).$$

Moreover, $\exists C = C_\alpha$, $|\alpha| \geq 0$ such that

$$|D^\alpha y_\pm(s,t)| < C \cdot e^{-\delta s}, \qquad s \in (0, l_i).$$

We now study the behavior of the (a,θ)-component of $\tilde{u}, \tilde{w}$ and $\tilde{v}_i$.

We have shown before that when $s > N$ or $s < -N$, for u and w, and $s \in (-l_i, l_i)$ for v_i, $|D(u(s,t) - x(t))| < \epsilon$. Since $|Dy(s,t)| < \epsilon$, this implies that $|\partial_t \theta - 1| = |\partial_t \theta - \partial_t x(t)| < \epsilon$ and $|\partial_s \theta| = |\partial_s \theta - \partial_s(x(t))| < \epsilon$. Let $\mathcal{P} : U \subset M \to \mathbf{R}^1 \times S^1 = \{(a,\theta)\}$ be the projection of the (a,y,θ)-coordinate chart U to (a,θ)-coordinate chart $\mathbf{R}^1 \times S^1$ given by $(a,y,\theta) \to (a,\theta)$. Then $\underline{u} = \mathcal{P} \circ \tilde{u}$, $\underline{w} = \mathcal{P} \circ \tilde{w}$ and $\underline{v}_i = \mathcal{P} \circ \tilde{v}_i$ are local diffeomorphisms from $[-N, +\infty) \times S^1$, $(-\infty, -N] \times S^1$ and $(-l_i, l_i) \times S^1$ to $\mathbf{R}^1 \times S^1$. Since

$$|\partial_s a(s,t) - 1| = |\partial_s(a(s,t) - cs)| < \epsilon$$

$|\partial_t a(s,t)| < \epsilon$ for these values of s in the above range

$$\begin{aligned} |a(s,t) - a(s_0,t_0)| &\geq |a(s,t) - a(s_0,t)| - |a(s_0,t) - a(s_0,t_0)| \\ &\geq \frac{1}{2}|s - s_0| - \epsilon. \end{aligned}$$

This implies that $\underline{u}$, $\underline{w}$ and $\underline{v}_i$ are proper.. Hence they are covering maps from open cylinders $(N, +\infty) \times S^1$, $(-\infty, -N)$ or $(-l_i, l_i) \times S^1$ to their images in $\mathbf{R} \times S^1$. Assume that the degree of the covering is m.

Let $\pi_m : \mathbf{R} \times S^1 \to \mathbf{R} \times S^1$ be the standard m-fold covering induced from the corresponding covering of S^1 to S^1. Write $\underline{u}$, $\underline{w}$ and $\underline{v}_i$ as (a,θ). We will study $\underline{v}_i$ first. We will only derive the equation for $\underline{v}_i = (a,\theta)$. The same formula is also applicable for $\underline{u}$ and $\underline{w}$.

Let $q_0 = \underline{v}_i(-l_i, 0) = (a_0, \theta_0)$ and $\tilde{q}_0 = (a_0, \tilde{\theta}_0) \in \pi_m^{-1}(q_0)$ with $\tilde{\theta}_0 \in [0,1)$ being the smallest of such $\tilde{\theta}_0$. Define $\underline{V}_i$ to be the unique lifting of $\underline{v}_i$ sending $(-l_i, 0)$ to $\tilde{q}_0$. We drop the subscript of $\underline{V}_i$ from now on. Then $\underline{V} : (-l_i, l_i) \times S^1 \to \mathbf{R} \times S^1$ is an embedding. Note that the length of the image of a-projection of $\underline{V}(\{l_i\} \times S^1)$ and $\underline{V}(\{l_i\}) \times S^1)$ is less than ϵ. Hence the image of $\underline{V}$ in $\mathbf{R} \times S^1$ is almost a standard cylinder of the form $[0, L] \times S^1$. We want to prove that $|L - 2l_i|$ is uniformly bounded and tends to zero when i and N tends to infinity.

Since π_m preserves a-length, this also implies the corresponding statement for $\underline{v}_i$.

To this end, define the complex structure $\underline{i} = \underline{i}(s,t)$ on the image of $\underline{V}$ by the identification:

$$\underline{i}(s,t) = dV(s,t) \circ i \circ \{d\underline{V}(s,t)\}^{-1} : T_{\underline{V}(s,t)}(\mathbf{R}^1 \times S^1) \to T_{\underline{V}(s,t)}(\mathbf{R}^1 \times S^1).$$

Then $\underline{V}$ is $(i, \underline{i})$-holomorphic, i.e.

$$d\underline{V} \circ i = \underline{i}(\underline{V}) \cdot d\underline{V}.$$

Equivalently,

$$\frac{\partial \underline{V}}{\partial s} + \underline{i}(s,t)\frac{\partial \underline{V}}{\partial t} = \frac{\partial \underline{V}}{\partial s} + \underline{i}(\underline{V})\frac{\partial \underline{V}}{\partial t} = 0.$$

Switch to $v_{i,+}$ or $v_{i,-}$ and consider the corresponding $\underline{v}_{i,+}, \underline{v}_{i,-}$ and $\underline{V}_+, \underline{V}_-$ and associated $\underline{i}(s,t)$. By abusing our notations, we will still use $\underline{i}(s,t)$ to denote the complex structure in these cases.

Lemma 4.10 *For $s \in (-l_i, l_i)$, there exists a constant $C = C_\alpha$ independent of i, such that $|D^\alpha(\underline{i}(s,t) - i)| < C_\alpha \cdot e^{-\delta s}$.*

Proof:

Since π_m is a local diffeomorphism, if we define $\underline{I} = \underline{I}(s,t) : T_{\underline{v}(s,t)}(\mathbf{R}^1 \times S^1) \to T_{\underline{v}(s,t)}(\mathbf{R}^1 \times S^1)$ by the formula: $d\underline{v}(s,t) \circ i \circ \{d\underline{v}(s,t)\}^{-1}$, then $\underline{i}(s,t) = d\pi_m^{-1} \circ \underline{I}(s,t) \circ d\pi_m$. Therefore, we only need to prove the corresponding statement for $\underline{I}(s,t)$. Now $\underline{v}$ is (i, I)- holomorphic, i.e. $d\underline{v}(s,t) \circ i = I(s,t) \circ d\underline{v}(s,t)$. In terms of the basis $\frac{\partial \underline{v}}{\partial s}(s,t)$, $\frac{\partial \underline{v}}{\partial t}(s,t)$,

$$I(s,t)(\frac{\partial \underline{v}}{\partial s}, \frac{\partial \underline{v}}{\partial t}) = (\frac{\partial \underline{v}}{\partial s}, \frac{\partial \underline{v}}{\partial t}) \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We need to find the expressions for $\frac{\partial \underline{v}}{\partial s}$ and $\frac{\partial \underline{v}}{\partial t}$ in terms of $(\frac{\partial}{\partial a}, \frac{\partial}{\partial \theta})$.

Sublemma

$$(\frac{\partial \underline{v}}{\partial s}, \frac{\partial \underline{v}}{\partial t}) = (\frac{\partial}{\partial a}, \frac{\partial}{\partial \theta}) \cdot \left\{ \begin{pmatrix} a_s & -a_t \\ a_t & a_s \end{pmatrix} + O(e^{-\delta s}) \right\}.$$

Proof:

$$\begin{aligned} \frac{\partial \underline{v}}{\partial s} &= d\mathcal{P} \circ d\tilde{v}_i(\frac{\partial}{\partial s}) \\ &= d\mathcal{P}(a_s \frac{\partial}{\partial a} + \{(v_i)_s - \lambda(v_i)_s X_\lambda\} + \lambda(v_i)_s X_\lambda). \end{aligned}$$

Let $(v_i)_s - \lambda(v_i) X_\lambda(v_i)_s = \Sigma_{k=1}^{2n} c_k \frac{\partial}{\partial y_k} + c_0 \frac{\partial}{\partial \theta}$. Then

$$\begin{aligned} (v_i)_s - \lambda(v_i) X_\lambda(v_i)_s &= \Sigma_{k=1}^{2n} c_k e_k \\ &= \Sigma_{k=1}^{2n} c_k \frac{\partial}{\partial y_k} + \Sigma_{k=1}^{2n} c_k \cdot (e_k - \frac{\partial}{\partial y_k}). \end{aligned}$$

Now c_k is uniformly bounded and $|e_k(u(s,t)-\frac{\partial}{\partial y_k})|<C\cdot|y(s,t)|<C\cdot e^{-\delta s}$. Similarly,

$$\lambda(v_i)_s X_\lambda=\lambda(v_i)_s(X_\lambda-\frac{\partial}{\partial\theta})+\lambda(v_i)\cdot\frac{\partial}{\partial\theta},$$

and

$$|X_\lambda(u(s,t))-\frac{\partial}{\partial\theta})|<c\cdot|y(s,t)|<c\cdot e^{-\delta s}.$$

This implies that

$$\begin{cases}\frac{\partial \underline{v}}{\partial s}=a_s\frac{\partial}{\partial a}+\lambda\{(v_i)_s\}\frac{\partial}{\partial\theta}+O(e^{-\delta s})\\ \frac{\partial \underline{v}}{\partial t}=a_t\frac{\partial}{\partial a}+\lambda\{(v_i)_t\}\frac{\partial}{\partial\theta}+O(e^{-\delta s}).\end{cases}$$

Now $\lambda\{(v_i)_s\}=-a_t$ and $\lambda(v_i)_t=a_s$. The conclusion follows.

QED

Let

$$A=\begin{pmatrix}a_s & -a_t\\ a_t & a_s\end{pmatrix},\ \text{and } O=O(e^{-\delta s}).$$

Then in terms of the basis $(\frac{\partial}{\partial s},\frac{\partial}{\partial t})$:

$$\underline{I}(s,t)=(A+O)\cdot J_0(A+O)^{-1}.$$

Since $AJ_0=J_0A$, $\underline{I}(s,t)=J_0+O(e^{-\delta s})$. This proves the lemma for $\alpha=0$. The general case with $|\alpha|\geq 1$ can be proved similarly.

QED

Still work with $\tilde{v}_{i,+}$ and the corresponding $\underline{V}$. Now $\underline{V}:(0,l_i)\times S^1\to\mathbf{R}^1\times S^1$. By $\mathbf{R}$-translation, we may assume that $\underline{V}(0,0)=(0,\theta_0)$. Note that $\theta_0\to 0$ when $i\to\infty$. Consider the unique lifting of $\underline{V}$ from the universal covering $(0,l_i)\times\mathbf{R}^1$ of $(0,l_i)\times S^1$ to the universal covering $\mathbf{R}^1\times\mathbf{R}^1$ of $\mathbf{R}^1\times S^1$, which sends $(0,0)$ to $(0,\theta_0)$, $0\leq\theta_0\leq 1$. We still denote it by $\underline{V}$. Then $(\underline{V}-Id):(0,l_i)\times\mathbf{R}^1\to\mathbf{R}^1\times\mathbf{R}^1$, and since both $\underline{V}$ and Id commutes with deck transformations induced by $\theta\to\theta+1$,, $\underline{V}-Id$ is periodic on the second factor of $(0,l_i)\times\mathbf{R}^1$ of period 1. Let $\Phi=\underline{V}-Id:(0,l_i)\times S^1\to\mathbf{R}^2$. Then

$$\frac{\partial\Phi}{\partial s}=\frac{\partial\underline{V}}{\partial s}-\frac{\partial(Id)}{\partial s}=-\{i+O(e^{-s})\}\frac{\partial\underline{V}}{\partial t}-i\frac{\partial(Id)}{\partial t}.$$

Since $\frac{\partial\underline{V}}{\partial t}$ is bounded, we have

$$\frac{\partial\Phi}{\partial s}+i\frac{\partial\Phi}{\partial t}+O(e^{-\delta s})=0.$$

Proposition 4.1 *$|\Phi(s,t)|<C$ for all $s\in(0,l_i)$, where C is bounded by the initial value of $O(e^{-\delta s})$, $|\Phi(0)|$ and $|\frac{\partial}{\partial t}\Phi|$. All of them tend to zero as i and N tends to infinity.*

Proof:

Let $\phi(s) = \int_{S^1} \Phi(s,t)dt$. Then

$$\frac{d\phi}{ds} = -\int_{S^1} O(e^{-\delta s})dt = f(s)(= O(e^{-\delta s})).$$

Hence $\phi(s) = \phi(0) + \int_0^s f(\tau)d\tau$. If $|f(s)| < d \cdot e^{-\delta s}$, $s \in (0, l_i)$, then $|\int_0^s f(\tau)d\tau| < \frac{d}{\delta}$.

Now let $\Psi(s,t) = \Phi(s,t) - \phi(s)$. Then

$$\int_{S^1} \Psi(s,t)dt = \phi(s) - \phi(s) = 0.$$

Let $C_1 = \max|\frac{\partial}{\partial t}\Phi(s,t)| = \max|\frac{\partial}{\partial t}\Psi(s,t)|$. Clearly $|\Psi(s,t)| < 2C_1$. Hence

$$|\Phi(s,t)| < |\Psi(\theta)(s,t)| + |\phi(s)| < |\phi(0)| + \frac{d}{\delta} + 2C_1.$$

QED

We reamrk that this proposition is the precise statement we mentioned before on the behavior of the "connecting neck" along the non-compact a- direction, which is used in the previous section to justify why it is possible to get the compactification of the moduli space without introducing the unstable trivial connecting maps.

For $\tilde{u}$ and $\tilde{w}$, we get more. We only prove the result for $\tilde{u}$. Define $\underline{u}$, $\underline{U}$ and $\Phi = \underline{U} - Id : (N, -\infty) \times S^1 \to \mathbf{R}^2$ as above. We have

$$\frac{\partial \Phi}{\partial s} + i\frac{\partial \Phi}{\partial t} + O(e^{-\delta(s-N)}) = 0.$$

Let $O(e^{-\delta(s-N)} = f(s,t)$. We identify the image $\mathbf{R}^2$ of Φ and f with $\mathbf{C}$. Then the standard complex structure $i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ on $\mathbf{R}^2$ is identified with the multiplication by imaginary number i. Fix s, let

$$\begin{aligned} \Phi(s,t) &= \Sigma_{n \in Z} \phi_n(s) e^{int} \\ f(s,t) &= \Sigma_{n \in Z} f_n(s) e^{int} \end{aligned}$$

be the Fourier expansion of $\Phi(s,-)$ and $f(s,-)$. Then $\phi_n' - n\phi_n + f_n = 0, n \in Z$. Note that $|f_n(s)| < C \cdot e^{-\delta(s-N)}$. In particular, when $n = 0$, $\phi_0'(0) = f_0(s)$. Hence

$$\begin{aligned} \phi_0(s) &= \phi_0(N) + \int_N^s f_0 \\ &= \phi_0(N) + \int_N^\infty f_0 - \int_s^\infty f_0 \\ &= s_0 - \int_s^\infty f_0, \end{aligned}$$

where $s_0 = \phi_0(N) + \int_N^\infty f_0$ is a constant. Now

$$|\int_s^\infty f_0(s)ds| < C \cdot \int_s^\infty e^{-\delta(\tau-N)}d\tau < \frac{C}{\delta}e^{-\delta(s-N)}.$$

Hence $|\phi_0(s) - s_0| < C_1 \cdot e^{-\delta(s-N)}$. Now let $\Psi(s,t) = \Phi(s,t) - \phi_0(s)$ and $r(s,t) = f(s,t) - f_0(s)$. Then

$$\frac{\partial\Psi}{\partial s} + i\frac{\partial\Psi}{\partial t} + r(s,t) = 0.$$

Note

$$< i\frac{\partial\Psi}{\partial t}, i\frac{\partial\Psi}{\partial t} >=< \Sigma_{n\neq 0} n\phi_n(s)\cdot e^{int}, \Sigma_{n\neq 0} n\phi_n(s)\cdot e^{ins} >\geq \Sigma_{n\neq 0}\phi_n^2(s) =< \Psi, \Psi > .$$

Define $g(s) = \frac{1}{2} < \Psi(s), \Psi(s) >$. Then

$$\begin{aligned} g''(s) &= < \Psi'(s), \Psi'(s) > + < \Psi''(s), \Psi(s) > \\ &= 2 < i\Psi_t, i\Psi_t > + < r, r > + < i\Psi_t, r > + < r, i\Psi_t > - < r_s, \Psi > \\ &\geq 2\|\Psi\|(\|\Psi\| - \|r\| - \|r_s\|). \end{aligned}$$

Note that both $|r(s,t)|$ and $|r_s(s,t)| \leq C \cdot e^{-\delta(s-N)}$. Let $h(s) = \|r(s,t)\| + \|r_s(s,t)\| \leq 2C \cdot e^{-\delta(s-N)}$. Then, if $\|\Psi\|(s) > 2h(s)$, we have

$$g'(s) \geq 2\|\Psi\|(\|\Psi\| - \frac{1}{2}\|\Psi\|) \geq g(s).$$

Now the set $P = \{s \quad | \quad \|\Psi\|(s) > 2h(s)\}$ is open and is a countable union of (s_i, s_{i+1}) such that $\|\Psi\|(s_i) = 2h(s_i)$ and $\|\Psi\|(s_{i+1}) = 2h(s_{i+1})$. Assume that $\delta > 1$, then the argument before to prove Lemma **??** implies that, for $s \in (s_i, s_{i+1})$, $g(s) \leq 4 \cdot C \cdot e^{-\delta(s-N)}$. On the other hand, if $s \notin P$, then $\|\Psi\|(s) \leq 2h(s) = 4C \cdot e^{-\delta(s-N)}$.

We conclude that $g(s) \leq C_1 \cdot e^{-\delta(s-N)}$. As before, applying elliptic estimate to get higher order estimate, we get

Proposition 4.2 *Let $\tilde{u}(s,t) = (u(s,t), a(s,t))$ be a $\tilde{J}$-holomorphic map such that $\lim_{s\to\infty} u(s,t) = x(t)$ of a closed orbit of λ-period c. Let $u(s,t) = (y(s,t), \theta(s,t))$ in the local (y,θ)-coordinate near $x(t)$. Then there exist positive constants $N, C = C_\alpha$ and δ such that*

$$\begin{aligned} |D^\alpha y(s,t)| &< C_\alpha \cdot e^{-\delta s}, \\ |D^\alpha\{(a(s,t), \theta(s,t)) - (cs + d_1, ct + d_2)\}| &< C_\alpha \cdot e^{-\delta s} \end{aligned}$$

for some suitable constants $d_1 \in \mathbf{R}$ and $d_2 \in (0, \tau)$, where τ is the minimal period of $x(t)$.

5 Some possible applications

The following are some immediate possible applications. Here we will only briefly indicate the reasons for these applications and refer the reader to the forth coming papers on each of these topics.

• **(A) Index homology in contact geometry**:

We have already outlined the index homology in contact geometry by using the moduli space of connecting pseudo-holomorphic maps. Note that the moduli space of connecting maps used here has an one dimensional symmetry of S^1-rotations. At the same time, their asymptotic ends of closed orbits also have the S^1-symmetry. It is possible to remove the symmetry by using the connecting $\tilde{J} = \tilde{J}_t, t \in S^1$ holomorphic maps with t dependent $\tilde{J}$. This will lead to a special Bott-type index homology, even the contact structure is generic.

• **(B) Additive quantum homology in contact geometry**:

The index homology we defined is an analogy of the usual Floer homology in symplectic geometry. We now outline a quantum homology in contact geometry by a different way to use the moduli space.

Let $M_a \subset \tilde{M} = M \times \mathbf{R}^1$ be the section $M \times \{a\}$ in $\tilde{M}$. Given any singular chain α in $M = M_a$ consider the moduli space $\tilde{\mathcal{M}}(x_-, x_+; \alpha, M_a)/S^1$, which is a subset of $\tilde{\mathcal{M}}(x_-, x_+)/S^1$ whose element u satisfies the condition that $u(z_0) \in \alpha(\in M_a)$. Now consider another marked point z_1 in u lying on a fixed marked line and define the obvious evaluation map $e_\alpha = e_{\alpha;a,b,z_1}$: $\cup_{x_-,x_+}\tilde{\mathcal{M}}(x_-, x_+; \alpha, M_a)/S^1 \to M_b$. Note that each element u with the two marked points without any non-compact symmetry anymore.

The intuition here is that by letting a being very negative, and b very positive, we flow the singular chain α lying almost in the negative end to get a collection of singular chains almost in the positive end.

We now define additive quantum homology by defining the chain complex generated by singular chains α in M with the boundary $D(\alpha) = \partial(\alpha) \pm e_\alpha$, where $\partial(\alpha)$ is just the usual boundary map of singular homology. By the property of the moduli space established in this paper and [L1], we have $D^2 = 0$. One can show that the homology so defined is independent of the choices involved and is an invariant of M.

• **(C) Gromov-Witten invariants in contact geometry and ring structure in the index cohomology** :

Whence the index homology is defined, we can define G-W invariant in exact the same fashion as the G-W invariant in the usual quantum homology and Floer homology. Namely given closed orbits $x_- = (x_{1,-}, \cdots, x_{k,-})$ and $x_+ = (x_{1,+}, \cdots, x_{l,+})$, we define G-W invariant $\Psi_{k,l}(x_-, x_+)$ by counting J-homomorphic map u in $\tilde{M}$ from the domain S^2 with $k + l$ punctures such that along k negative ends u approaches to x_- and along l positive ends u approaches to x_+.

As in the usual GW-invariant in quantum and Floer homology, one can show that the invariant so defined at chain level descends to the homology.

By using the invariant $\Psi_{2,1}$, one can define a ring structure in the index cohomology, which can be thought as a quantum product for the contact manifold.

One can also extend to definition of G-W invariants by introducing another set of marked points $z = (z_1, \cdots, z_n), z_i \in S^2$, and require that $u(z_i) \in C_i$ of some prescribed cycles in $\tilde{M}$.

Using the special case of three marked point invariants with only one z, we get an action of $H_*(M)$ on the index homology, i.e. the index homology is a module over $H_*(M)$.

There are obvious generalization of theses constructions, such as higher genus G-W invariants, coupling with gravity and so on.

Note that the product structure should be thought as an essential part of the structure of these index homologies as there are many cases where the additive index homology is infinitely generated.

• **(D) Relative quantum homology**:

Give a compact symplectic manifold (P, ω) with a contact boundary, let the boundary be M with the compatible contact structure $\lambda = i_X\omega$, where X is the contact field i.e. $\mathcal{L}_X\omega = \omega$. We now glue $\tilde{M}$ to P along the boundary. By using a suitable choice of ϕ mentioned before, we get a new symplectic manifold with a cylindrical end.

The chain complex of the quantum homology of a symplectic manifold P with contact boundary M is generated by the pair (α, β) where α is a singular chain in P and β is a singular chain in $\tilde{M}$. The boundary operator $D = (D_1, D_2)$. Here $D_2(\beta)$ is defined same as the one in (A) above. $D_1(\alpha) = \partial(\alpha) \pm e_\alpha$. The definition of e_α here is also similar to the one in (B). But we use the moduli space of J-holomorphic maps, the domain of whose elements is $\mathbf{C}$ being treated as a half sphere with an half infinite cylinder attached, to flow the singular chain α in P to get a collection of singular chains e_α in M. Again we have $D^2 = 0$. Now there is an obvious embedding of the chain complex of the quantum homology of M defined in (B) to the chain complex we just defined. We define the chain complex of the relative quantum homology as the quotient of this pair.

Note that unlike (B) above, in the case that the contact boundary M of P is concave, we may not be able to get a desired uniform energy bound. In this case we need some extra assumption such as ω is exact.

Note that there are some obvious algebraic constructions related to these chain complexes, such as the induced long exact sequences related these three homologies and Mayer-Vietoris sequence of these homologies. More general, assume that we can decompose a compact symplectic manifold in sequence of increasing symplectic sub-manifolds with (convex) contact type boundaries, we can associate the sequence a filtration of chain complexes defined above. Then there is a associated spectral sequence associated to the filtration.

It is an interesting question to study further these algebriac constructions to incorperate the multiplicative structures and to study their relation to the quantum homology of a symplectic manifold. It seems that this will give a new way to compute quantum homology of a symplectic manifold.

• **(E) Bott-type index homology, S^1-invariant contact manifold and Weinstein conjecture**:

We have assumed so far that the contact form λ is generic so that the set of closed orbits is discrete. We can relax this condition by only requiring that λ is of Bott-type. Then the set of closed orbits decomposes into an union of different components, each being a manifold. Note that the period of any element in a component is the same by Stokes theorem. In the symplectic case, in this situation, Ruan and Tian developed a Bott-type Floer homology. One can develop a similar construction in this case. As remarked in A, we have two different versions of the Bott-type homology.

One of our motivation to consider Bott-type index homology is to answer the question that if the index homology so defined is always trivial.

By using the Bott-type index homology, one can compute the index homology when the contact manifold appears as a regular zero locus of a local Hamiltonian function on some symplectic manifold, which generates a local S^1-action.

For simplicity, let (P, ω) be a compact symplectic manifold with a S^1 Hamiltonian action generated by a Hamiltonian function H. Assume that a is a regular value of H. Let $P^a = H^{-1}(a)$ and $P_a = P^a/S^1$. Under some assumption, P^a is a contact manifold whose contact structure is specified by ω. In fact the contact structure on P^a can be chosen to be S^1-invariant. We define a S^1-invariant contact form as follows. Note that $P^{(a-\epsilon,a+\epsilon)}$ is a S^1 bundle over $P_{(a-\epsilon,a+\epsilon)}$, where $P^{(a-\epsilon,a+\epsilon)} = H^{-1}((a-\epsilon, a+\epsilon))$ and $P_{(a-\epsilon,a+\epsilon)} = H^{-1}((a-\epsilon, a+\epsilon))/S^1$. Chose a connection. We can lift any vector field X, which is transversal to P_a in $P_{(a-\epsilon,a+\epsilon)}$ to an S^1-equivariant vector field $\tilde{X}$. We define the S^1-invariant contact form $\lambda = i_{\tilde{X}}\omega$. By adjusting X, we may assume that $\lambda(X_H) = 1$. That is λ the connection 1-form for the S^1-bundle. Hence, λ is a contact form if the curvature $d\lambda$ is positive. Now the set of closed orbits of the contact manifold (P^a, λ) of period 1 is just P_a and the images of these closed orbits foliated P^a itself. All other components of the set of closed orbits are just copies of this one according to different periods. We are in the situation of Bott-type index homology. The chain complex of Bott-type homology is generated by singular chains in some components of the set of closed orbits and the boundary map is the combination of the usual boundary map for singular homology together with a "connecting" map by using the connecting J-holomorphic maps between two components of the set of closed orbits to flow the singular chain. In our case, due to the extra S^1-symmetry in the moduli space of J-connecting maps, the second part, the part of the "connecting" map, of the boundary map has no contribution. Hence, the Bott-type index homology is just infinitely many copies of the usual homology of the symplectic quotient P_a. In view of the invariance of Bott-type index homology, this also compute the index homology for the contact structure. In particular, we proved the non-vanishing of index homology in this case.

As a corollary, we proved Weinstein conjecture for this case.

It would be interesting to study the relationship of the product structure in the contact manifold P^a with the quantum homology of its quotient, the symplectic manifold P_a.

• **(F) Gluing formula for G-W invariants**:

Give a compact symplectic manifold (P, ω), assume that there is a contact type hypersurface $M \subset P$ such that M cuts P into two pieces P_- and P_+ with the common boundary M. As in (D), we can glue $\tilde{M}$ to each of P_- and P_+ to form two non-compact symplectic manifolds P^- and P^+ with cylindrical ends. As in [LR], we can prove a gluing formula for G-W invariants, which relates the G-W invariants of P with the G-W invariants in P^+, P^- and $\tilde{M}$.

The idea is the following:

One first collect all J-holomorphic map u in P^+, P^- or $\tilde{M}$ with the property that u approaches to some of closed orbits lying on the ends of P^+, P^- or $\tilde{M}$ along its punctures, then select among them those u can be glued along those closed orbits.

Note that unlike in [LR], we do not require any local S^1 Hamiltonian action.

• **(G) Low dimensional contact manifold** A special feature of a three dimensional compact manifold is that it always has a contact structure. Hence the index homology and additive quantum homology is well-defined associated to the contact structure. It would be very interesting to investigate if the invariants we defined here are actually topological invariants. There are various different forms of this type of questions. In view of the work of Taubes on the relationship of the SW-invariants and GW-invariants, one may hope to get similar results for contact 3-fold and symplectic four manifold with contact type boundary. Our result should serve as one of the basis to formulate this type of results.

We make the following final remark. As we mentioned before, one of the main results of this paper and [L3] is about the virtual co-dimension of the boundary of the moduli space, which is the foundation of the applications outlined in this section. This result is the consequence of the compactness theorem proved in this paper and the index formula, which will be proved in [L3]. To obtain the result, the index formula we need here is different from the usual one appeared in Bott-type Floer homology due to the extra dimension of the asymptotic R^1-motion of a connecting pseudo-holomorphic maps along the ends (closed orbits).

On the other hand, the main body of this paper, the proof of the compactness theorem, is independent of the desired index formula. In fact, the new phenomenon appeared in the bubbling described in Lemma 3.4 and the Definition 4.1 on equivalence of stable maps concerning how to count symmetries in target already opens the door for various possible applications.

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